ON RECENT DEVELOPMENTS IN TECHNIQUES OF PLANE ELASTIC PROBLEMS

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Dedicated

To

My Beloved

Father Mr. (Late)

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CERTIFICATE

Certified that ***Mr. Sarfaraz*** carried out his dissertation work entitled ***"ON RECENT DEVELOPMENTS IN TECHNIQUES OF PLANE ELASTIC PROBLEMS"*** under the supervision of the undersigned and the work is found suitable for submission for the award of Master of Philosophy degree.

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CHAPTER-1

PRELIMINARY

1.1 INTRODUCTION

The foundations of the mathematical theory of elasticity were laid at the beginning of the XIXth century by L. M. H. Navier, M. V. Ostrogradskii, A. L. Cauchy, S. D. Poisson, B. P. E. Clapeyron and others. A general formulation of the problems of the theory of elasticity in displacements is due to G. Lame, while such a formulation in stresses was for the first time given by E. Beltrami.

The one who introduced the first formulation with the. aid of a potential function for a problem of the theory of elasticity was however the English astronomer George-Biddell Airy (1801-1892), in two successive papers [5], [4], published in 1862 and 1863 and devoted to the study of the straight beam. Since the "plane case" brings great simplifications of calculation as compared to the general three-dimensional problem and since the plane problems are of a particular interest from the practical and technical standpoints, Airy's initial results were followed in the last hundred years by extremely numerous works in which a great variety of mathematical methods were used. As there are thousands of such works, practically it is impossible to review them or to present a complete bibliography; therefore we shall restrict ourselves to mentioning the works we consider more important (formulation of problems, fundamental works, followed by subsequent investigations, etc.), trying to emphasize various steps in the development of this problem as well as its future outcome. A more detailed bibliography may be found in the monographs dedicated to this problem and mentioned at the end of this paper, as well as in some standard books. [15], [16], [36].

a) From the mechanical standpoint two plane problems are to be distinguished, which may be considered as classical: (i) the case of a plane state of stress, corresponding to a plane plate of constant thickness, free of load on the parallel faces and acted upon on its boundary by loads parallel to the middle plane and uniformly distributed upon the plate thickness; *(ii)* the case of a plane state of strain, corresponding to a long cylindrical body (theoretically infinite), acted upon on its lateral surface by a uniformly distributed load along the generatrix, without tangential component in the direction of the generatrix. The case of a quasi-plane state of stress (the z variable is neglected) or the case of a generalized plane state of stress (the mean state of plane stress) reduce to the classical case of a plane state of stress; a similar result is obtained in the case of a generalized plane state of strain (for which the displacement along the cylinder axis is a harmonic function).

b) Plane plates subjected to a plane state of stress are also known under the name of deep beams. A great number of construction elements are included in this class, such as: deep beams used for silos and single- and multi cell bunkers, deep beams without parallel boundaries (gables for bridges, launched beams, etc.), horizontal or vertical stiffening diaphragms for mean and tall buildings in seismic areas, short cantilevers for rolling bridges, frame corners, junction plates for metallic structures, various machine parts (hooks, connecting rods, etc), construction elements for ships, stiffening elements for aircraft wings, etc. Likewise, as examples of bodies subjected to a plane state of strain we cite: heavy dams, supporting walls, tubes, tunnels, factory chimneys, vaults, long plates, rolls for bridge supports, the half-space subjected to uniformly distributed loads in one direction (geotechnical problems), pressure tubes, etc. The great variety of the construction elements mentioned above shows, from the practical standpoint, the particular importance of the plane problem.

1.2 THE PLANE PROBLEM OF THE THEORY OF ELASTICITY

In the classical case of a plane state of stress, it is assumed that three components of the stress tensor are equal to zero

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0, \qquad (12.1a)$$

the remaining components being different from zero and independent of the *z* variable. Likewise, in the classical case of a plane state of strain, three components of the strain tensor are taken equal to zero

$$\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0, \qquad (12.1b)$$

the remaining components being different from zero and independent of *z*. If the stresses and strains may also depend on *z*, then we have a state of stress varying paiabolically upon the plate thickness in the case of a plane state of stress and linearly along the generatrix in the case of a plane state of strain [20], [67]. D. Morgenstern [89] has shown how the case of a plane state of stress may be rigorously obtained for a plane plate whose thickness tends to zero.

a) In the classical case, the two problems—distinct from the mechanical standpoint—lead to the same mathematical formulation. The equilibrium equations

$$\left.\begin{aligned}\frac{\partial\tau_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+X=0,\\ \frac{\partial\tau_x}{\partial x}+\frac{\partial\sigma_y}{\partial y}+Y=0,\end{aligned}\right] \tag{1.2.2}$$

where X, Y are the components of the body force, σ_x, σ_y the normal stresses and τ_{xy} the tangential stress, the relations between the linear strains ε_x, ε_y the angular strain γ_{xy} and the displacements u, v

$$\varepsilon_x=\frac{\partial_u}{\partial x},\quad \varepsilon_y=\frac{\partial_v}{\partial y}, \tag{1.2.3a}$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}, \tag{1.2.3b}$$

and Hooke's law (for linearly elastic homogeneous and isotropic bodies)

$$\varepsilon_x=\frac{1}{E}(\sigma_x-\mu\sigma_y), \tag{1.2.4a}$$

$$\varepsilon_y=\frac{1}{E}(\sigma_u-\mu\sigma_x) \tag{1.2.4b}$$

$$\gamma_{xy}=\frac{1}{G}\tau_{xy}$$

valid in the case of a plane state of stress, are given; here E is the longitudinal elasticity modulus, μ is Poisson's ratio and G is the transverse elasticity modulus given by

$$G = \frac{E}{2(1+\mu)}, \qquad (1.2.5)$$

b) In the case of a plane state of strain the same relations are used, in which the elastic constants are replaced by the generalized elastic constants

$$E_{\upsilon} = \frac{E}{1-\mu^2}, \quad \mu_{\upsilon} = \frac{\mu}{1-\mu} \qquad (1.2.6)$$

G.-B._Airy [5], [4] has shown that, in the absence of body forces, equations (2) may be satisfied if we take

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad (1.2.7a)$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \qquad (1.2.7b)$$

Where $F=F(x,y)$ is an arbitrary function, and gave several simple examples of calculation in polynomial form. J. C. Maxwell [78] has shown however that the deformation of the body must be also considered and introduced the strain continuity equation, written in stresses under the form

$$\Delta(\sigma_x + \sigma_y) = 0, \qquad (1.2.8)$$

where Δ is Laplace's operator. It results that the function *F*, named Airy's function, must be biharmonic

$$\Delta\Delta F(x, y) = 0 \qquad (1.2.9)$$

The stress function F assumes an interesting mechanical interpretation, particularly on the boundary. Thus, the tangential derivative $\delta F/\partial s$ and the normal derivative $\partial F/\delta n$ on the boundary are the shear force T (with contrary sign) and the axial force *N*, respectively, in a fictitious beam, which follows the boundary counterclockwise, leaving it on the left side, proceeding from an arbitrarily chosen fixed point; likewise, the function F on the boundary is the bending moment M in the same fictitious beam. In this manner, the first fundamental plane problem of the theory of elasticity (case in which the external loads acting on the boundary are given), for a simply connected region, in the absence of body forces, reduces—from the mathematical standpoint—to the determination of a biharmonic function $F=F(x,y)$ for which (i) the value of the function and the normal derivative (M and *N*), or (ii) the value of the normal derivative and of the tangential derivative (*N* and T), are known on the boundary. The first of these formulations is the most usual one. The second formulation is also known under the name of biharmonic fundamental problem and was examined from the mathematical standpoint particularly beginning from 1907, when it constituted the object of a premium of the Paris Academy of Sciences awarded to works by J. Hadamard, G. Lauricella [68] and A. Korn, dealing with a single connected finite region which satisfies certain general conditions. A more general study was carried out subsequently by S. L.

Sobolev [120]. From the physical viewpoint, the uniqueness of the solution is ensured by G. R. Kirchhoff's uniqueness theorem. Likewise, except for a first degree polynomial (three arbitrary constants), the stress function and its partial derivatives are uniquely determined throughout the domain if the external loads equilibrate on the boundary. J. H. Michell [82] has shown that this result remains valid even if the external loads reduce to a couple.

In the case of a multiconnected region distortions must be introduced and the general theory given by V. Volterra [135] must be employed. The state of stress in such a body depends, generally, on the elastic constants of the material and becomes independent of these constants only if the external loads equilibrate on each separate boundary [82]. L. N. G. Filon [28] has shown, however, that practically the influence of the elastic constants on the maximum stress is in general very small and may be neglected. Generally one may use a stress function of Airy type corresponding to certain conditions set 'into evidence by G. Grioli [48]. The doubly connected regions were particularly treated by W. Prager [104] and D. I. Sherman [114].

c) A formulation in displacements of the problem with the aid of a biharmonic function F= $F(x, y)$ is due to K.Marguerre and is of the form

$$\left.\begin{aligned} u &= -\frac{1+\mu}{1-\mu}\frac{\partial^2 F}{\partial x \partial y} \\ v &= +\frac{2}{1-\mu}\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2} \end{aligned}\right] \qquad (1.210)$$

or of a similar form, obtained by changing the role of the *x* and *y* variables. This formulation is useful for the second fundamental problem (conditions in displacements on the boundary) or in the case of the mixed problem (conditions in stresses on one side of the boundary and in displacements on the other side). In this case the function F loses the mechanical significance given previously.

1.3 METHODS OF COMPUTATION

The methods of calculation for the plane problem are very varied as they have developed in time, indicating thus the evolution of the problem from the mathematical standpoint.

As shown above, the first method presented consisted in the employing of real potentials (stress or displacement functions) and, consequently, in the examination of the bi-harmonic equation. An important part is played here by the representation given by E. Almansi [11]

$$F(x,y)= \Phi(x,y) + [a + bx + cy + d(x^2 + y^2)]\Psi(x,y), \quad (1.3.1)$$

where Φ and Ψ are harmonic functions and *a, b, c, d are* arbitrary constants (in order not to obtain a harmonic function, *b, c, d* can not simultaneously vanish). This representation permits one to study the structure of a biharmonic function and to construct such a function with the aid of two harmonic ones. An important study of biharmonic and polyharmonic functions was made by M. Nicolesco [97]. Various investigations have been made in this respect and other representations by real potentials were also suggested, such as for instance that given by L. Payne [103] for the elastic half-plane and the elastic strip.

a) The search for various forms of stress functions resulted in useful methods of computation for some particular domains. The indirect method of computation consists in assuming a certain state of stress in the interior of the body, which should fulfill the boundary conditions, and in verifying if all equations of the theory of elasticity are satisfied; contrarily, another stress function must be chosen and the computation repeated. Eventually, one may add to the results yielded by some approximate methods of calculation (for instance the results given by the strength of materials) or to the state of stress initially assumed, some correction functions of which must be determined; this is a semiindirect method of computation. The direct method consists in selecting a potential function (which should fulfill eventually certain odd-ness or evenness conditions with respect to the two variables or other conditions which could be initially imposed, function of the particular conditions of the problem) which should include certain arbitrary parameters; these parameters must be determined with the aid of the boundary conditions, If these parameters can be determined, fulfilling thus all the boundary conditions, the problem is solved. Contrarily, another potential function must be chosen, with the aid of which the calculation is to be performed (or to add a correction to the initially chosen function). Another aspect of the direct methods is that the problem can be approached through a general method (e.g., finite difference method, complex variable method, etc.), which may lead to a systematic calculation. Obviously, each of these methods is suited for various particular cases.

Mathematically, one may perform an exact calculation or an approximate one. The cases in which an exact calculation may lead to results are very rare, since the physical phenomena are generally complex and can be covered but approximately by various mathematical forms; therefore most calculation methods employed are approximate ones. The approximations may come either from the selection of the stress function or of the displacement function under a form which approximates best the real form (e.g., a trigonometric series) and which fulfills completely the biharmonic condition and the boundary conditions, or from the search for a function which should fulfill approximately some of these conditions (the finite difference method, certain variational methods, the method of conditions at points on the boundary, etc.); we note that happens rather frequently if both kinds of approximations are employed.

b) Other approximate methods of calculation which have begun to be frequently used are the operational methods. Within these methods, the functions intervening in the partial derivative equations of the problem, called primitive functions, transform into other functions, called images of the first functions; The property is reciprocal, since the image of a primitive function is also the primitive function. In this manner, the partial derivative equations of the problem transform" into algebraic equations which may be solved by usual methods of calculation; after that one returns to the primitive functions which correspond to the solution of the problem. Within the operational methods, one may use various types of transforms, i.e. Fourier, Laplace (in Cartesian coordinates), Mellin (in polar coordinates), etc.

For general problems related to these transforms and for applications to the plane problem of the theory of elasticity, reference can be made to [117], [133]. Within these calculation methods, it is useful to use generalized functions (in the sense of the distribution theory) in order that the case of concentrated loads be also included. In the case of a finite domain one may use such finite transformations as lead to series developments.

c) It is often convenient to use the potential function under the form of a Fourier series or under the form of a Fourier integral, with a sequence of undetermined coefficients or with an unknown function, which are determined with the aid of the boundary conditions; this is in fact but a direct application of the Fourier transform, without passing through all stages of computation. This idea was used for the first time by M. Ribiere [106] L. N. G. Filon [27] and A. Timpe [129] for the straight beam and elastic strip and by J. H. Michell [81] for the circular domain, and constitutes the second important stage in the development of the calculation methods of the plane problem.

These ideas were developed for a great number of domains interesting from the practical standpoint and led either to the solution of an infinite system of linear algebraic equations with an infinity of unknowns (for which one is confronted with problems of regularity, complete regularity, etc), or to the solution of an equation or system of Fredholm-type integral equations of the second kind, on an infinite interval (for which one is confronted with problems of existence and uniqueness, e.g. case of generalized functions); eventually, integral equations may be degenerate, leading thus also to the1 solution of a system of linear algebraic equations. The operational methods mentioned above constitute in fact a new and much more general aspect of the first investigations carried out at the beginning of this century. It is worth noting that the Fourier series and integrals may be constructed by proceeding from the particular integrals of the harmonic equation of the form (method of the separation of variables)

$$e^{\alpha x+\beta y} \qquad (\alpha^2+\beta^2=0)$$

using representation (11) and the principle of the superposition of effects.

d) The third important stage in the development of the calculation methods for the plane problem is constituted by the introduction of complex variable functions. Proceeding from the representation given by A. E. H. Love [67] for the components of the displacement vector, G. V. Kolosov [60] gave a representation of the complex displacement (in the absence of body forces) of the form

$$2G(u+iv)=\frac{3-\mu}{1+\mu}\varphi(z)-\overline{z\varphi'(z)}-\overline{\psi(z)} \qquad (1.3.2)$$

where $\varphi(z)$ and $\psi(z)$ are holomorphic functions of $z = x + iy$, the prime sign indicates the derivation and the upper line the complex conjugate function. Likewise, the state of stress will be given by

$$\sigma_x + \sigma_y = 4\mathrm{Re}[\Phi(z)] \qquad (1.3.3)$$

$$\sigma_y - \sigma_x + 2\, i\tau_{xy} = 2[z\Phi'(z) + \psi(z)]$$

where the following notations

$$\Phi(z) = \varphi(z),\ \psi(z) = \dot{\psi}(z)$$

(1.3.4)

were used. This representation by complex potentials is closely related to that by real potentials, if one considers Goursat's [39] formula for biharmonic functions

$$F(x,y) = \mathrm{Re}[\bar{z}(z) + R(z)] \qquad (1.3.5)$$

where X(z) and K(z) are holomorphic functions. The functions Φ, ψ, φ, ψ are uniquely determined for a certain state of stress and a certain state of strain, except *Ci,* where C is a real constant—for the function $\Phi(z)$—except *(Ciz* + y), where $\gamma = \alpha + i\beta$ is an arbitrary complex constant—for the function $\varphi(z)$—and except the complex constant $\gamma' = \alpha' + i\beta'$ for the function $\psi(z)$. The influence of these constants on the state of strain consists in the introduction of a displacement and a rotation corresponding to the motion of a rigid body.

Setting the boundary conditions, the three fundamental problems of the theory of elasticity can be reduced to problems of complex variable functions. In order to formulate these conditions, we shall denote by X(t) limit toward which the function *X(z)* tends when the point z in the interior of the domain tends toward the point *t* on the boundary. For the first fundamental problem, the boundary conditions may be set under the form

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = i\int_{t_0}^{t} (p_{nx} + ip_{ny})ds \qquad (1.3.8)$$

where (p_{nx}, p_{ny}) are the components of the external load, acting on an element of boundary of external normal *n*. Likewise, in the case of the second fundamental problem we have

$$\frac{3-\mu}{1+\mu}\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = 2G(u_n + iv_n), \qquad (1.3.9)$$

where (U_n, V_n) *ace* the displacements corresponding to the element of boundary considered above. As conditions (18) and (19) are similar, it results that the methods of calculation applicable in the case of the two fundamental problems are similar. When solving effectively the problems, it is practical to impose that the functions $\phi(z)$, $\phi'(z)$ and ϕ (z) could each be prolonged continuously at all points on the boundary. The corresponding solution of the problem is termed regular solution.

D) With the aid of the method of complex variable functions, outlined above, certain interesting problems of deep beams with various contours were solved. N. I. Muskhelishvili [90], [92] carried out particularly important studies creating in this respect, together with his followers, a real school in the USSR.

Apart from Muskhelishvili's treatise [94] these investigations are presented in various books of a general character, devoted to the theory of elasticity, e.g. [43], [98], [122]. We mention particularly the book by I. Babuska, K. Rektorys and F. Vycichlo. Some incipient forms of this method were given by L, FoppI [31]. The Kolosov-Muskhelishvili method was refound subsequently by A. C. Stevenson [124], [123]. L. M. Milne-Thomson [88] published, under an original presentation, an interesting monograph on this problem in which many boundary-value problems were solved. S. M. Belonosov also published a monograph on these problems in which he referred to simply- and doubly-connected regions. But the first formulation of the plane problem of the theory of elasticity with the aid of complex variable functions was given by S. A. Chaplygin; the work appeared only after the death of the author. It is to be mentioned that the problem may be also formulated by using the concept of areolar derivative, introduced by D. Pompeiu; such a formulation was made by N. Teodorescu [126]. Another manner of presenting the problem and systematizing the calculation is due to R. Legendre [65] and Cl. Mathurin. Formulations based on employing hypercomplex variable functions were also given, such as that suggested by L. Sobrero [121].

A powerful mathematical tool of this method is the con-formal mapping of the domain considered previously. Performing also the transformation for external loads, we transform the original problem to a simpler problem. After obtaining the solution of this problem, the inverse conformal mapping is made, which brings us back to the initial problem. For performing this transformation, power series or Fourier series under complex form as well as the representation on the interior (finite domain) or exterior (infinite domain) of a circle or the representation on a circular annulus (for doubly connected regions) are used. Some interesting results are given in [24]. The mapping functions were used, for the first time, in the form of polynomials by E. Almansi [11]; results in this direction for domains acted upon by concentrated forces have been given by L. Martini [75].'

The application of the properties of Cauchy-type integrals to the solving of the plane boundary problems is also due to N. I. Muskhelishvili. We can mention thus the case of domains represented on a circle with the aid of rational functions; these results lead also to an approximate solution for arbitrary domains. Another method of computation consists in reducing the problem to the problem of linear conjugation of the boundary-value problems (known as Riemann's problem or Hilbert's problem); in this case interesting solutions for slit domains as well as for mixed problems of the theory of elasticity can be given (eventually for contact problems).

e) With the aid of the method of Cauchy-type integrals the plane problem can be reduced to integral equations which can be set under the form of Fredholm-type integral equations of the second kind, requiring the conformal mapping of the considered domain on a circle; the method is useful for singly-connected regions. S. G. Mikhlin [85] modified this method by using complex Green functions with a logarithmic singularity such that the method be also applicable to multi-connected regions. Results obtained in this respect may be found in his treatise 83]. The integral equations suggested bv X. I. Muskhelishvili [91], [90] may be applied to domains bounded by a more general type of boundary, if the integrals in these equations are considered Stieltjes integrals or Racion-Carleman integrals; such a procedure was applied by L. G. Magnaradze for domains bounded by contours with angular points. These types of equations may be successfully used for the numerical solution of various plane boundary problems. Another important formulation under the form of integral equations of complex functions was given by D. I. Sherman [114], [115], a formulation similar to that obtained by G.

Lauricella with real functions. The Lauricella-Sherman equations are useful for the first two fundamental problems and led to numerous applications. In the case of the mixed problem, Sherman [115] obtained some singular integral equations which were subsequently studied by G. F. Mandzhavidze [72]. Ch. Massonet and L. Beskin also established some integral equations of the problem.

From the above considerations it. is seen that the representation of the solution of the plane problem of the theory of elasticity with the aid of real or complex potentials permits the construction of a perfectly well-framed theory of this problem, simpler and more complete than for any other computation method.

F) B. Sen [110] has solved ..certain problems through a procedure of the superposition of effects. G. A. Grinberg, INT. JST. Lebedev and Ya. S. Uflyand tried to give a general form for the potential function. I. Babuska [12] has continued the investigations for certain particular domains. Generally, the approximate methods are very varied; for instance, W. Valentin [134] presented a method for calculating a deep beam by strips.

The methods presented lead themselves generally to interesting graphical interpretations; for instance, P. Dubas' [25] method admits such .a graphical calculation since the funicular polygons may be easily constructed. We recall the mechanical interpretation of the stress function, the stresses in a normal section of a bar being obtainable by graphical means. Ch. Massenet's [76] graphical method can be somehow situated midway between the analytical methods and the experimental ones, proceeding from certain known states of stress and using an adequate summation term. Likewise, H. Poncin discussed the case in which certain deformations are known on the body contour (deformations imposed or measured on a certain deep beam or model) and showed how these magnitudes can be also calculated in the interior by a procedure of functional extension.

G) The possibility of employing electronic computers is also to be mentioned. Such computers may be used for performing various intermediate computations (e.g. solving of systems of linear algebraic equations which intervene in the method of networks, in the method of setting the boundary conditions by points, etc.), or for accomplishing a program according to a certain algorithm of calculation. It is to be mentioned that apart from digital computers one may use' analog computers, performing for instance a finite difference integration upon one direction and a continuous integration upon the other direction, with the aid of the differential analyzer.

H) K. Wieghardt [138] observed in 1908 the analogy between the biharmonic equation verified by Airy's function and the equation of plane thin plates, subjected to bending, in the absence of a load normal to the middle plane. The stress function *F may* be thus obtained experimentally, on a model, as the deformed middle surface of a thin plane plate, free of load, but subject to certain deformations on the boundary (corresponding to the boundary conditions in the plane problem). The normal stresses will be proportional to the curvatures of this surface and the tangential stress will be proportional to the geodesic torsion. This analogy was extended by R. D. Mindlin to multiconnected regions for the action of body forces and temperature variations. Worth noting is that tests were carried out in this respect by P. Dantu [23] who employed an original procedure. A. S. Kalmanok [56] used this analogy for the analytic calculation of a deep beam with rectangular boundary. Likewise, an interesting analogy of the plane problem of the theory of elasticity with the problem of the slow motion of a viscous fluid was given by J. N. Goodier [37], [38] in 1934.

1.4. BOUNDARY-VALUE PROBLEMS

We shall refer now to the three fundamental problems for single-connected, elastically linear, homogeneous and isotropic bodies, in the case of small deformations, subjected to the action of loads applied on the contour and to body forces. The considered domains can be generally classified according to the system of coordinates employed.

Thus, an important number of studies were carried out in orthogonal Cartesian coordinates, particularly for a rectangular domain or .for infinite domains obtained from this one by throwing to infinity one, two, three, or all its sides (theaction of the body forces will be disregarded at the beginning). For bibliographical data, reference can be made to [126], the methods of calculation employed being particularly the elementary ones or the Fourier series or integrals. A great number of studies were devoted to the problem of the straight beam of rectangular cross-section, which may be considered subjected to a state of plane stress. The results obtained may be compared to those given by the methods of the strength of materials, establishing thus the applicability limit of these latter methods. Usually in such a case global conditions are set (for stresses in the cross-section) on two of the rectangle sides. The correct setting of the boundary conditions for an elastic rectangle leads to the solution of an infinite system of linear algebraic equations with an infinity of unknowns. The first fundamental problem and the mixed problem are particularly of interest (*some of the sides being built-in or* with imposed displacements, elastically built-in sides, etc.). The infinite domains mentioned above may be similarly studied by using stress functions from the same family [126]. In the case of the elastic half-strip one obtains an infinite system of linear algebraic equations and in the case of the quarter of elastic plane a system of Fredholm-type integral equations of the second kind. For the elastic half-plane and elastic strip the final results can be explicitly written. It is worth noting that for each of these problems various methods of calculation from those indicated above were used and can be used, beside the method of real potentials (*considered in our presentation*) and the method of complex potentials.

a) A study in oblique Cartesian coordinates permits the results mentioned above to be easily transformed for an elastic parallelogram or for infinite plane domains obtained from this one by throwing to infinity one or several of its sides. By analogy with the quarter of elastic plane, results are obtained for the elastic wedge.

b) For domains bounded by circle arcs and polar radii the polar coordinates are very useful. Interesting studies in this conjunction were made by J. H. Michell [81] who also established basic principles for the transformation by inversion; he examined the case of the circular disk as well as that of an elastic wedge under the action of concentrated forces at the vertex. We mention the works of A. V. Gadolin and H. S. Golovin concerning the plane deformation of thick cylinders.

c) In such a system of coordinates, B. G. Galerkin gave interesting results for the infinite elastic trapezium. The circle sector and the circular annulus sector, the elastic wedge and in particular the elastic half-plane are examined. Apart from the monographs indicated above, it is very difficult to indicate a bibliography on this subject (over one hundred papers were written only for the elastic wedge in various cases of support and load). We mention only the paradox observed recently in conjunction with S. D. Carothers [21], problem (the elastic wedge acted upon by a concentrated moment at the vertex) by E. Sternberg and W. T. Koiter, according to which the classical results depend on the wedge vertex angle. A very detailed study in this respect was published by H. Neuber [95], specialist in the problem of local states of stress. Proceeding from the case of the elastic wedge and elastic half-plane subjected to concentrated loads on the boundary line, approximate studies in polar coordinates can be made for arbitrary domains acted upon by concentrated loads on the boundary.

d) The study of the infinite elastic plane subjected to-the action of concentrated loads (concentrated force, concentrated moment) can be made in various ways; in this case the use of the distribution theory is of interest as it permits a correct representation of such a load by generalized functions. We mention the problem of passing from a concentrated force applied at a point to the same force applied at a very neighboring point. We have thus the notion of directed moment (couple of forces which depends on the direction of forces) and the notion of a system of two contrary forces equal in magnitude and applied at very neighboring points; if such forces act on the contour of a circle of small radius, one obtains a nucleus of elastic deformation. In the case of any finite or infinite domain one may use the results obtained for the elastic plane and the principle of the superposition of effects, the problem being reduced to the study of the respective domain acted upon by certain conventional loads.

e) The problems examined and mentioned above have all the same characteristic: the respective domains are bounded by lines of coordinates in the system of coordinates employed, such that on the boundary one of the coordinates becomes constant (simplifying thus the setting of the boundary conditions and the solving of the problem). This suggests the use of a system of curvilinear coordinates, function of the given domain, such that its contour be formed by lines of coordinates. A study was thus made and a formulation of the problem in arbitrary' curvilinear coordinates was given; as particular cases one may use isogonal, orthogonal, isothermic coordinates as well as arbitrary rectilinear coordinates, translation coordinates, and so on. We mention that up to the present, mostly isothermic coordinates were used.

1.5. SPECIAL PLANE PROBLEMS

The calculation methods employed for the classical plane problem of the theory of elasticity can be extended to a great number of such problems in which other types of external loads intervene or for which the constitutive law of the material differs from that corresponding to the linear elastic, homogeneous and isotropic case. Firstly, we shall mention the case of the multiconnected domains; in such a case the structure of the potential functions undergoes certain modifications through the appearance of terms corresponding to the number of holes. For real potentials, the problem was investigated by G. Grioli and for complex potentials by N. I. Muskhelishvili [94]; an interesting synthesis was given by Cr. Teodosiu.

This problem is of interest for investigating stress concentrations in bodies with holes. The case of the elastic plane and half-plane with one or two holes of various forms, e.g., circular, elliptical, etc., was thus examined. The case of an infinite row of such holes (the problem, reduced to an integral equation, was examined by S. G. Mikhlin [84] and the difficult case of a finite number of holes were also examined. These domains were generally considered as acted upon at infinity by a uniform stress or subject to their own weight (the problem is of importance in geotechnics). The problem becomes more difficult in the case of finite domains with interior holes. The method of complex variable functions was successfully applied to many problems by G. N. Savin. Different problems concerning concentration of stress are to be found in [95]. Another important case is that in which the holes reduce to a straight or curved line and cracks occur. The problem is approached either by tending to limit a hole, or by special methods suitable to this case. Interesting results in this direction have been given by G. I. Barenblat.

a) Recently, increasing attention has been given to the quasi-plane problems both for obtaining a more accurate solution, as mentioned at point 2.1., and for approximating the state of stress and strain in more complex construction elements such as deep beams of variable thickness.

b) A special problem, of particular importance in practice, is the contact problem. Mathematically, the problem reduces to the integration of integral or integro-differential equations. Special attention was given to the contact of two cylindrical bodies bounded by any lateral surfaces along a common generatrix, considering that both bodies are deformable or that one of them plays the part of a rigid punch. One problem of particular interest is the action of a rigid punch on an elastic half-plane (the punch may be rectangular, or it may be a stamp of various forms, i.e., circular, parabolic, etc). In conjunction with these problems one may consult the monographs by L. A. Galin [34] and I. Ya. Shtaerman [116].

An interesting problem, initiated and examined by A. Sig-norini, is that of an elastic body continuously supported on a rigid half-plane; under the action of external loads the elastic body deforms, some points of its boundary being separated from the separation line of the elastic half-plane. Since one cannot know *a priori* which points are included in the first category and which points in the second one, this is an ambiguous boundary-value problem which aroused numerous studies out of which we cite those by G. Fichera (under print in Italian publications).

c) The plane problems assume an interesting formulation in the limit case of incompressible bodies (Poisson's ratio $\nu = 1/2$). It is worth noting that this case may constitute a first approximation of computation for the general problem (compressible body), which may be reduced to the solving of a sequence of incompressible problems. Investigations in this respect were carried out by J. Golecki.

d) The case of linearly elastic, homogeneous and anisotropic bodies has stimulated a growing interest due to the fact that the real bodies and the construction materials employed enjoy such properties. After the last century, controversy concerning the elastic constants came to an end by the unanimous recognition of 21 distinct elastic constants and by the accepting of the energy considerations made by G. Green, but a long time elapsed before concrete boundary-value problems could be solved. The case of transversally isotropic bodies and that of orthotropic bodies were particularly discussed (practically, the case of stiffened deep beams can be approximate with the case of orthotropic deep beams of constant thickness). Obviously, all the problems mentioned in this section were also set for anisbtropic bodies. Results of a general character, formulations of the problem (by similarity with formulations in real and complex potentials for isotropic bodies) and solutions of boundary-value problems are given in [43], [86], [126]; we mention particularly the monographs

[66].

In some simpler cases the problem may be formally reduced to problems corresponding to isotropic bodies. Most times, however, qualitatively differing results are obtained. Mathematically this is explained by the fact that the fourth-order partial derivative equation which governs the phenomenon is no longer biharmonic (it may be reduced to a biharmonic equation only in particular cases). In this case the representation formula (11) is no longer valid and one may try eventually a representation of the solution with the aid of T. Boggio's [18] theorem, as shown in [126]; we note that such a *representation is no longer valid* when, *at limit*, we pass to the isotropic body.

e) As concerns the nonhompgeneity, distinction must be made between continuous and discontinuous nonhomogeneity (stratified bodies). The plane problem for isotropic bodies with continuous nonhomogeneity (characterized by elastic co-efficients functions of point) was formulated by real potentials [126] and complex potentials [84] and led generally to linear equations with variable coefficients; it is to be noticed that these equations may be integrated by a method of successive approximations, a series of classical plane problems with conventional boundary *conditions* being thus *solved* for the same domain. Thus, the problem of elastic wedge was examined by an inverse method by S. G. Lekhnitskii [66]. W. Olszak and J. Rychlewski tried to determine the non-homogeneity corresponding to given elementary states of stress. We mention that the problem equations have constant coefficients only if the elastic coefficients vary exponentially from point to point.

As concerns the stratified bodies, the following cases were examined: the infinite plane consisting of two half-planes with different elastic properties, the domain consisting of an elastic strip and the elastic half-plane (which corresponds also from the technical standpoint to a deep beam on an elastic beam [126], the domain consisting of two elastic strips, etc. From the seismic standpoint, important problems are to be examined in conjunction with the propagation and reflection of seismic waves in such media, e.g. Love-type waves.

The general case of nonhomogeneous anisotropic bodies was also considered. It is to be mentioned that up to the present time the concept of nonhomogeneity was not sufficiently well defined; indeed, by a transformation into arbitrary curvilinear coordinates, proceeding from an anisotropic and homogeneous body, one may obtain a form of Hooke's law, corresponding to a curvilinearly anisotropic nonhomogeneous body. It was tried to define the notion by assuming that a body remains nonhomogeneous in any system of coordinates employed; the problem remained, however, open.

f) Special problems confront *us* in the case of loose media, case in which random variables can be successfully introduced.

Important problems are also to be confronted in the case of deformable solids characterized by more complicated constitutive laws. We mention thus the case of nonlinearly elastic bodies from the physical point, the cases of hypoelastic and hyperelastic bodies, etc., for which a lot of work is to be done as concerns the plane problem.

Likewise, many results obtained in the classical case as well as through the methods of calculation indicated above can be extended to linearly rheologic bodies (viscoelastic bodies). Moreover, for certain types of viscoelastic bodies the results obtained in the classical case can be readily adapted. For elastoplastic bodies the problem differs slightly, according to the adopted theory of plasticity. We mention that within the theory of dislocations (which tends to reduce the problems of the theory of plasticity to problems of the theory of elasticity by the introduction of the concept of dislocation density, characterizing the plastic properties of the body) one may extend the calculation methods of the classical case; besides, up to the present few investigations have been carried out in this respect (for bibliographical data see.

g) The general problem of finite elastic deformations is substantially simplified in the plane case. However, up to the present few studies were made in this respect. For general bibliography and results see [43], [98].

h) A great amount of attention was paid lately to complex problems in which the mechanical phenomena are to other physical phenomena. Thus, in thermoelasti equation of heat propagation is associated with the equation of the theory of elasticity. Most studies were carried the case in which, by knowing the temperature v throughout the considered domain, one passes to the of the elasticity problem, formally the problem being to that in which arbitrary body forces appear; the i case involves *greater* difficulties of calculation.

Of particular theoretical interest is the case of equations (in the equation of heat propagation the in of the elastic solid compressibility also intervenes); equations can no longer be separated one must find complicated representation by real potentials. It i noting, however, that practically in most cases this accurate treatment of the problem has very slight influence compared to the approximate case (uncoupled equation. The thermoelastic problems are of interest both for elastic, isotropic and homogeneous bodies and for anis or nonhomogeneous problems [126],. For bibliogr; data and results obtained in this direction see the monographs so far published [19], [35], [74].

i) Another discipline which developed in the la years, mainly in Poland, was magnetoelasticity (the equation of the theory of elasticity are associated to Maxwell's of the electromagnetic field). In the linearized rather large number of studies on the propagation of *mi* elastic waves are done [140].

When body moments and couple stresses appear (dui electromagnetic field), the stress tensor becomes asymmetric so we are confronted with a study of asymmetric elasticity. Results in this respect, as well as for the case in v local asymmetry occurs due to concentrated moments body external surface, are given by G. Grioli[48].

j) Apart from the *disciplines mentioned* above, have developed more or less *recently, investigations an in complex domains* such as, for instance, thermoelas *magnetoelasticity*, etc. The plane problem of magnetot elasticity was also formulated [99]. All these facts men's endeavour to study problems related to the phenomena of nature in their complexity, observing the interdepen that exists between them.

1.5.1 ABSTRACT OF THE DISSERTATION

The chapter I of the present dissertation gives the general introduction and brief history of the mathematical theory of elasticity. The dissertation mainly deals with the plane problem of eleasticity and so an attempt has been made to give the preliminary information of this type of problems. Some very fundamental equations have been introduced.

In chapter II, the classical method of complex variable has been discussed. The governing equations of equilibrium in stress components have been used for the determination of stresses. Formulation of the problem has been done with the help of complex variable technique and the equations have also been converted in terms of complex variable z. As conformal mapping plays a very important role in plane problem of elasticity, the expression for components of stress and strain have been derived in terms of ζ where $z=m(\zeta)$ is the mapping function. Several first fundamental problems have been discussed and their results analyzed.

Chapter - III, a direct method ahs been discussed which was dev eloped by Sen [109]. This method, though not a general one, sometimes gives results very conveniently. The method involves selection of some function from institution but its practical usefulness can not be ignored. In the later part of this chapter integral transform techniques have been discussed. A problem of initial stress solved by Kurashige [61] has been considered. The application of the developed method by various workers like

Ahmad and Ali has been studied.

Chapter IV, deals with the techniques of integral transform which are found to be very convenient for solving indentation and crack problems. It contains some important integral transforms which are used in solid mechanics. Using this technique, a crack problem of an initially stressed neo-Hookean solid has been solved by Hankel transform. The components of incremental displacement and incremental stress are found out and their variations are studied graphically.

Chapter V, some punch problem have been considered within the frame work of complex variable technique developed by Milne-Thomson [86], Green and and Zerna [42] etc. Later on some modern techniques which are very useful for problem of elasticity have been considered. One of the methods i.e. finite element method has been discussed. In connection with the finite element method. several variational principle have also been discussed. At the end of the chapter, a problem of inclusion solved by Ishikawa and Kohno in 1993 has been discussed.

1.6.1 CONCLUSIONS

The study of the classical plane problems presents itself an undeniable interest. A great number of result useful in practice and theoretically of interest have been obtained up to the present. The importance of these s consists however in the ideas and lines of investigation suggest for unclassical plane problems, as shown in th ceding section. Most investigations carried out in his respect generalized the results obtained previously, showing also an eventual correction of these results.

a) Considering the fact that in the plane case important simplifications of calculus appear it is practical that v problems be first examined in such a case. This also indications of the possibilities of solving spatial pro For instance, the problem of the elastic parallelepiped that of the infinite domains obtained from this one by throwing to infinity one, two or several of its faces can be s *similarly as the problem of the elastic* rectangle and infinite plane domains obtained from this one by throw infinity one or several of its sides.

Likewise, classical photoelastic studies led to the dimensional photoelasticity.

All the above considerations show that the classic; nonclassical plane problems are real, much investigation maining to be done in this direction.

Chapter-2

CHAPTER – II

COMPLEX VARIABLE METHODS

2.1 MATHEMATICAL PRELIMINARIES:

The equation of motion in Cartesian coordinates (x, y, z) are given by

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_x}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}=x_1\\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}=x_2\\ \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \tau_z}{\partial z}=x_3\end{aligned}\right] \qquad (2.1.1)$$

Where x_1, x_2, x_3 are components of body forces σ_x, σ_y, σ_z the normal stresses and τ_{yx} etc. are the shearing stresses.

In the present chapter the main concern of the plane elastic system in which their exists a plane such that the stress tensor is the same to all material points of any normal to this plane as at the material point in which the normal meets the plane. Thus all derivatives with respect to z become zero.

Hence equations (2.1.1) become

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}=x_1\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=x_2\\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}=x_3\end{aligned}\right] \qquad (2.1.2)$$

It is further assumed that

$$\tau_{yz} = \tau_{zx} = 0 \qquad (2.1.3)$$

which makes (2) reduces to

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}=x_1\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=x_2\end{aligned}\right] \qquad (2.1.4)$$

It has been found that shearing tensor is symmetric i.e.

$$\tau_{xy} = \tau_{yx} \qquad (2.1.5)$$

Hence equation (4) are ultimately reduced to

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=x_1\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=x_2\end{aligned}\right] \qquad (2.1.6)$$

These are the governing equations of the plane elastic systems. Sicne these equations do not involve the z-coordinate, henceforth z will be used to denote the complex variable x+iy and their should be no confusion. Using

$$z = x+iy, \quad \bar{z}=x-iy \qquad (2.1.7)$$

we have

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial z}+\frac{\partial}{\partial z},\ \frac{\partial}{\partial y}=i\frac{\partial}{\partial z}-i\frac{\partial}{\partial z} \tag{2.1.8}$$

$$2\frac{\partial}{\partial x}=\frac{\partial}{\partial x}-i\frac{\partial}{\partial y},\ 2\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}+i\frac{\partial}{\partial y} \tag{2.1.9}$$

Using (2.1.8) and (2.1.9), we get from (2.1.6)

$$\frac{\partial}{\partial z}(\sigma_x+\sigma_y)-\frac{\partial}{\partial \bar{z}}(\sigma_y-\sigma_x+2i\tau_{xy})x_1-ix_2$$

or

$$\frac{\partial\theta}{\partial z}-\frac{\partial\Phi}{\partial z}=x_1-ix_2 \tag{2.1.10}$$

where

$$\theta=\sigma_x+\sigma_y,\ \Phi=\sigma_y+\sigma_x+2i\tau_{xy} \tag{2.1.11}$$

are called "fundamental stress combinations" and were introduced by Kolosov [58].

In the absence of any body forces, equations (2.1.10) are

$$\frac{\partial\theta}{\partial z}-\frac{\partial\Phi}{\partial \bar{z}}=0 \tag{2.1.12}$$

Equation (2.1.12) is identically satisfied if we take

$$\theta=4\frac{\partial^2 x}{\partial z\partial \bar{z}},\ \Phi=4\frac{\partial^2 x}{\partial z^2}=0 \tag{2.1.13}$$

Where $\chi(x,y)$ is a function of x and y and therefore of z and $\bar{z}$. The function $\chi(x,y)$ is known as 'Airy's stress function.

Again since, $x=\frac{1}{2}(z+\bar{z}),\quad y=-\frac{1}{2}i(z-\bar{z})$ (2.1.14)

thus in terms of fundamental stress combination $\odot$and Φ the stress components are given as follows:

$$\left.\begin{aligned}\sigma_x &= \frac{1}{2} \geq \odot - \frac{1}{4}(\Phi + \overline{\Phi}) \\ \sigma_y &= \frac{1}{2} \odot - \frac{1}{4}(\Phi + \overline{\Phi}) \\ \tau_{xy} &= -\frac{1}{4} i(\Phi - \overline{\Phi})\end{aligned}\right] \tag{2.1.15}$$

Similarly, in absence of body forces, equations (2.1.6)

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0\end{aligned}\right] \tag{2.1.16}$$

are identically satisfied, if we take

$$\left.\begin{aligned}\sigma_x &= \frac{\partial^2 x}{\partial y^2}, \ \sigma_y = \frac{\partial^2 x}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 x}{\partial x \partial y}\end{aligned}\right] \tag{2.1.17}$$

were $\chi(x,y)$ is the Airy's stress function. Thus we see that the solution of the plane elastic problems is reduced to determination of some potential functions like fundamental stress combinations or Airy's stress function.

2.2 PLANE DEFORMATION

A deformation is said to be in a state of 'plane deformation" if one of the principal direction of deformation is the same at every point of material and particles which occupy planes perpendicular to the fixed principal direction before the deformation remain on the same plane after the deformation.

Let u, v, w be the components of displacement of any point (x, y, R), than ignoring the rigid body displacement of the material, we must have

$$w = 0, \quad e_{zz} = 0 \tag{2.2.1}$$

Hence, we get

$$\left.\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x},\ e_{yy} = \frac{\partial v}{\partial y} \\ e_{xy} &= \frac{1}{2}\left[\frac{\partial v}{\partial x}, + \frac{\partial u}{\partial y}\right] \end{aligned}\right] \tag{2.2.2}$$

$$e_{xz} = e_{yz} = e_{zz} = 0 \tag{2.2.3}$$

The equation of compatibility in the case of plane deformation is

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \tag{2.2.4}$$

In terms of complex variable z, equation (2.2.4), can be written as

$$\frac{\partial^2 \bar{g}}{\partial z^2} + \frac{\partial^2 g}{\partial \bar{z}^2} = 2\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0 \tag{2.2.5}$$

where f and g are given by

$$f = e_{xx} + e_{yy}, \quad g = e_{yy} - e_{xx} + 2ie_{xy} \tag{2.2.6}$$

So far we have ignored the body forces. However considering the body forces let F_x, F_y be the components of the body forces which can be derived from a potential function V. Then we have

$$F_x - iF_y = \frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y} = 2\frac{\partial v}{\partial z} \qquad (2.2.7)$$

Following Milne-Themson [88] the Airy's stress function χ and potential function V are related by

$$\nabla^4 \chi + \frac{1-2\mu}{1-\mu}\nabla^2 V = 0 \qquad (2.2.8)$$

$$\left[\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

where μ is the Poisson ratio.

The case of generalized plane stress can be obtained from equation (2.2.8) by replacing μ by μ' where μ' is given by

$$(1-\mu')(1+\mu) = 1 \qquad (2.2.9)$$

Putting this value of μ' in equation (2.2.8), we get,

$$\nabla^4 \chi + \upsilon\, \nabla^2 V = 0 \qquad (2.2.10)$$

Where,

$$\left.\begin{array}{l} \upsilon = 2\dfrac{\alpha - 1}{\alpha + 1} \\ \alpha = 3 - \mu \ \textit{for plane deformation} \\ \alpha = \dfrac{3-\mu}{1+\mu} \ \textit{for generalized plane stress} \end{array}\right] \qquad (2.2.11)$$

Let us further suppose

$$\left.\begin{array}{ll} & V = \nabla^2 U \\ or & V = 4\dfrac{\partial^2 U}{\partial z \partial \bar{z}} \end{array}\right] \qquad (2.2.12)$$

Equation (2.2.10), then reduced

$$\left.\begin{aligned} &\nabla^4 = (\chi + \upsilon U) = 0 \\ \text{or} \qquad &16\frac{\partial^4(\chi + \upsilon U)}{\partial z^2 \partial \bar{z}^2} = 0 \end{aligned}\right] \qquad (2.2.13)$$

Thus we see that the function $\chi+\upsilon U$ is a plane biharmonic function. From equation (2.2.13) we see that $\chi+\upsilon U$ is a real valued plane harmonic function and so we can write

$$\nabla^2(\chi + \upsilon U) = W(z) + \overline{W}(\bar{z}) \qquad (2.2.14)$$

or

$$4\frac{\partial^2(\chi + \upsilon U))}{\partial z \partial \bar{z}} = W(z) + \overline{W}(\bar{z}) \qquad (2.2.15)$$

Integrating (2.2.15) with respect to $\bar{z}$, we get

$$4\frac{\partial(\chi + \upsilon U))}{\partial z} = \bar{z}\, w(z) + \int \overline{w}(\bar{z})d\bar{z} + w(z) \qquad (2.2.16)$$

where w(z) is the constant of integration.

Integrating (2.2.16) again with respect to z, we get

$$4(\chi + \upsilon U) = \bar{z} \int w(z)dz + z \int \overline{w}(\bar{z})dz + \int w(z)dz + \int \overline{w}(\bar{z})dz \quad (2.2.17)$$

The constant of integration in (2.2.17) is taken in the above form so as to make the right hand side real valued. However, in absence of body forces, U=0 and w get

$$4(\chi) = \bar{z} \int w(z)dz + z \int \overline{w}(\bar{z})d\bar{z} + \int w(z)dz + \int \overline{w}(\bar{z})d\bar{z} \qquad (2.2.18)$$

The complex displacement, in absence of body forces, ignoring the rigid body displacement is given by Milne-Thomson [88] as

$$4\mu(u + iv) = \alpha \int w(z)dz - z\overline{w}(\bar{z}) - \int \overline{w}(\bar{z})dz \qquad (2.2.19)$$

The functions W(z) and w(z) are called complex stresses.

2.3 COMPLEX STRESSES IN THE ISOTROPIC CASE

The material of the body is supposed to be isotropic. The Airy's stress function ζ is given by equation (2.2.18) when there is no body forces. Using equations (2.1.13), (2.1.15) and (2.1.18) the stresses are given by

$$\sigma_x + \sigma_y = \Theta = W(z) + \overline{W}(\bar{z}) \qquad (2.3.1)$$

$$\sigma_Y - \sigma_Y + 2i\tau_{xy} = \Phi = \bar{z}W'(z)w(z) \qquad (2.3.2)$$

The functions W(z) and w(z) are called complex stresses. From the above equations it is clear that they have the physical dimension of stress. Thus to solve a plane problem of elasticity is to determine the complex stresses W(z)and w(z)

2.4 COMFORMAL MAPPING

The results of some problems of a circular region are known. The conformal mapping provides a very powerful tool to solve a plane problem by mapping the boundary of the region under consideration into a circle .We can consider the case of a disc or that of a hole with a curvilinear boundary in an infinite plate .Let C, the boundary of the disc, or that of a hole in an infinite plate , be transformed on to the circumference $\Gamma$of the circle $|\zeta|$=a by the mapping function

$$z = m(\xi),\ z = x + iy,\ \xi = e^{\xi + i\eta} \qquad (2.4.1)$$

The radius of circle is giver by a=$e^{\zeta o}$, where the curve C is given by

$\zeta=\zeta_o$ (constant).

The stress components in the ζ-plane are given by

$$2(\sigma_\zeta + i\tau_{\zeta\eta}) = \odot \;- \Phi \frac{\overline{\zeta}\,\overline{m'(\zeta)}}{\zeta\; m'(\zeta)} \qquad (2.4.2)$$

$$2(\sigma_\eta + i\tau_{\zeta\eta}) = \odot \;+ \overline{\Phi} \frac{\overline{\zeta}\,\overline{m'(\zeta)}}{\zeta\; m'(\zeta)} \qquad (2.4.3)$$

where (2.4.2) is used for the members of family $\zeta=$ constant and (2.4.3) for those of $\eta=$ constant.

2.5. STRESSES IN ζ-PLANE

We consider the case of isotropic material which undergoes infinitesimal elastic deformation. Let C be the boundary of the material. The sense of description of C, so that the material lies in the left hand region denoted by L, is taken to be positive. The region not occupied by the material will be denoted by R. Let mapping function (2.4.1) maps the curve

C:$\zeta=\zeta_o$ into the circumference of the circle $\Gamma:|\zeta|=e^{\zeta o} = a$. It is further assumed that L of C is mapped into the interior or exterior of the circle Γ according as the material is inside 9 a disc) or out side C(a hole).

For the simplicity of the description, we denote the complex stresses by $W_0(z)$ and $w_0(z)$. So that we have

$$\odot = \overline{W_0(z)} + \overline{w_0}(\overline{z}), \overline{\Phi} = z\overline{w}_0'(\overline{z}) + w_0(\overline{z}) \qquad (2.5.1)$$

If we eliminate z, using (2.4.1), we get

$$\left.\begin{aligned} W_0(z) &= W_0[m(\zeta)] = W(\zeta), \quad say \\ w_0(z) &= w_0[m(\zeta)] = w(\zeta), \quad say \end{aligned}\right] \qquad (2.5.2)$$

The derivative of $w_o(z)$ is given by

$$w_o'(z) = \frac{dw_o(z)}{dz} = \frac{dw(\zeta)}{dz} = \frac{dw(\zeta)}{d\zeta}\cdot\frac{d\zeta}{dz} = \frac{w'(\zeta)}{m'(\zeta)} \qquad (2.5.3)$$

So that the complex stresses in the ζ-plane are given by

$$\theta = w(\zeta) + \overline{w}(\overline{\zeta}),\ \overline{\Phi} = \frac{m(\zeta)}{\overline{m'}(\overline{\zeta})}\overline{w}'(\zeta) + \overline{w}(\overline{\zeta}) \qquad (2.5.4)$$

Therefore the stress components are given by

$$2(\sigma_\zeta + i\tau_{\zeta\eta}) = w(\zeta) + \overline{w}(\overline{\zeta}), \frac{\overline{\zeta}m(\zeta)}{\zeta m'(\zeta)}\overline{w}'(\overline{\zeta}) + \frac{\overline{\zeta}\,\overline{m}'(\overline{\zeta})}{\zeta m'(\zeta)}\overline{w}'(\overline{\zeta}) \qquad (2.5.5)$$

$$2(\sigma_\eta - i\tau_{\eta\zeta}) = w(\zeta) + \overline{w}(\overline{\zeta}), \frac{\overline{\zeta}m(\zeta)}{\zeta m'(\zeta)}\overline{w}'(\zeta) + \frac{\overline{\zeta}\,\overline{m}'(\overline{\zeta})}{\zeta m'(\zeta)}\overline{w}'(\zeta) \qquad (2.5.6)$$

Similarly, differentiating equation (2.2.19) w.r.t. ζ the components of displacement will be given by

$$4\mu\frac{\partial}{\partial\eta}(u + iv) = [\alpha w(\zeta) - \overline{W}(\overline{\zeta})]i\zeta m'(\zeta) + [m(\zeta)\overline{w}'(\overline{\zeta}) +$$

$$+\overline{m}'(\overline{\zeta})\overline{w}(\overline{\zeta})]i\overline{\zeta} \qquad (2.5.7)$$

2.6 FIRST FUNDAMENTAL PROBLEM:

Let the boundary c be loaded by the stress given by

$$\sigma_\zeta + i\tau_{\zeta\eta} = -\rho(\sigma) + is(\sigma) \qquad (2.6.1)$$

Where $\rho(\sigma)$ is the pressure and $s(\sigma)$ is shear at a point σ on the circle Γ in which c has mapped.

The complex stress $w(\zeta)$ is given by Milne-Thomson [88].

$$m'(\zeta)w(\zeta)=\frac{1}{2\pi}\int\frac{2[-\rho(\sigma)+is(\sigma)]m'(\sigma)d\sigma}{\sigma-\zeta}+\psi(\zeta) \qquad (2.6.2)$$

So for the complete solution of the problem, w(ζ) and w(ζ) should be known. However, we can get rid of w(ζ) and express the stresses and strains in terms of only one complex stress i.e.w. (ζ) in the following way:

We have

$$\begin{aligned}m'(\zeta)w(\zeta)+m'(\zeta)\overline{W}(\bar{\zeta})-\frac{\bar{\zeta}}{\zeta}m(\zeta)\overline{W}'(\bar{\zeta})-\\-\frac{\bar{\zeta}}{\zeta}\overline{m}'(\zeta)\overline{w}(\zeta)=2(\sigma_{\zeta}+i\tau_{\zeta\eta})m'(\zeta)\end{aligned} \qquad (2.6.3)$$

(ζ in L)

We make the continuation of (2.6.3) by writing zero for the stresses, a^2/ζ for $\bar{\zeta}$ to give the equation valid in R:

$$m'(\zeta)w(\zeta)=-m'(\zeta)\overline{W}\left(\frac{a^2}{\zeta}\right)+\left(\frac{a^2}{\zeta^2}\right)m(\zeta)\overline{w}'\left(\frac{a^2}{\zeta}\right)+\left(\frac{a^2}{\zeta^2}\right)\overline{m}'\left(\frac{a^2}{\zeta}\right)\overline{w}\left(\frac{a^2}{\zeta}\right)$$

(ζ in R) (2.6.4)

From (2.6.4) by writing $\bar{\zeta}$ for $\frac{a^2}{\zeta}$ for ζ and taking the complex conjugate, we get

$$\begin{aligned}m'(\zeta)w(\zeta)&=\frac{a^2}{\zeta^2}\overline{m}'\left(\frac{a^2}{\zeta}\right)\left[w(\zeta)+\overline{w}\left(\frac{a^2}{\zeta^2}\right)\right]-\overline{m}\left(\frac{a^2}{\zeta}\right)w'(\zeta)\\&=-\frac{d}{d\zeta}\left[\overline{m}\left(\frac{a^2}{\zeta}\right)w(\zeta)\right]+\left(\frac{a^2}{\zeta^2}\right)\overline{m}'\left(\frac{a^2}{\zeta}\right)\overline{w}\left(\frac{a^2}{\zeta}\right)\end{aligned} \qquad (2.6.5)$$

(ζ in L)

The function w(ζ) may be eliminated from (2.6.3) and (2.6.5). The form of the function ψ(ζ) in (2.6.2) is obtained by considering the singularities of m' (ζ) W(ζ) in region R. The function ψ(ζ) is holomorphic in L and the solution must be non-dislocational.

Several problems of various nature have been solved using the above technique . Recently Ahmad and All [8], solved the first fundamental problem of a disc in the form of Pascal's limacon.

The disc has been considered to be in equilibrium under the concentrated forces. The mapping function

$$\left.\begin{array}{l} z = m(\zeta) = C(\zeta + K\zeta^2) \\ (C > 0, \quad 0 \le K \le \frac{1}{2}, \quad z = re^{i\theta} \end{array}\right] \tag{2.6.6}$$

has been used to map the jimacon in the z-plane onto a unit circle Γ(ζ=o) in ζ-plane.

The form of the function ψ(ζ) is taken as

$$\psi(\zeta) = \alpha_o + \beta_o\zeta \tag{2.6.7}$$

The concentrated load has been supposed to be uniformly distributed in a small neibhourhood near the points. The case of concentrated load F has been obtained by a limiting process. Stresses were determined and the stress intensity factor studied graphically. The cases of a plate in the form of a cardiod and a circle have been obtained as special cases.

2.7 ECCENTRIC ANNULUS

Later on Ali [10] discussed the first fundamental problems of an eccentric annulus which is in equilibrium under two standard concentrated forces F and -F applied to the outer boundary at points where it has the maximum and minimum thickness. These points are given by A(a_1 ,0) and B(a_2 0) where

$a_1 = c\frac{\beta+1}{\beta-1}$, $a_2 = c\frac{\beta-1}{\beta+1}$ [see fig.1]. In the present case, the complex potential functions have been made analytically continuous across inner aid outer boundaries of the annul us. This gives the "compatibility Identity".

The transformation function

$$z = m(\zeta) = c + \frac{2c}{\zeta - 1}(c > 0) \tag{2.7.1}$$

has been used to map the eccentric circles c_1: $\zeta=\zeta_1$ and c_2: $\zeta=\zeta_2$ ($\zeta_2>\zeta_1$) in the z-plane onto concentric circles Γ_1 and Γ_2 or radii $\beta(= e^{\zeta}{}_1)$ and α $(=e^{\alpha}{}_2)$, $\alpha>\beta$ respectively in the ζ-plane. It is to be noted that the mapping function (2.7.1), turns the annulus inside out. The complex potential W(ζ) is given by

$$\begin{aligned} m'(\zeta)W(\zeta) = &\frac{1}{2\pi i}\int_{\Gamma_1}\frac{2[-\rho(\sigma)+is(\sigma)]m'(\sigma)}{\sigma-\zeta}d\sigma + \\ &\frac{1}{2\pi i}\int_{\Gamma_2}\frac{2[-\rho(\sigma)+is(\sigma)]m'(\sigma)}{\sigma-\zeta}d\sigma + \psi(\zeta) \end{aligned} \tag{2.7.2}$$

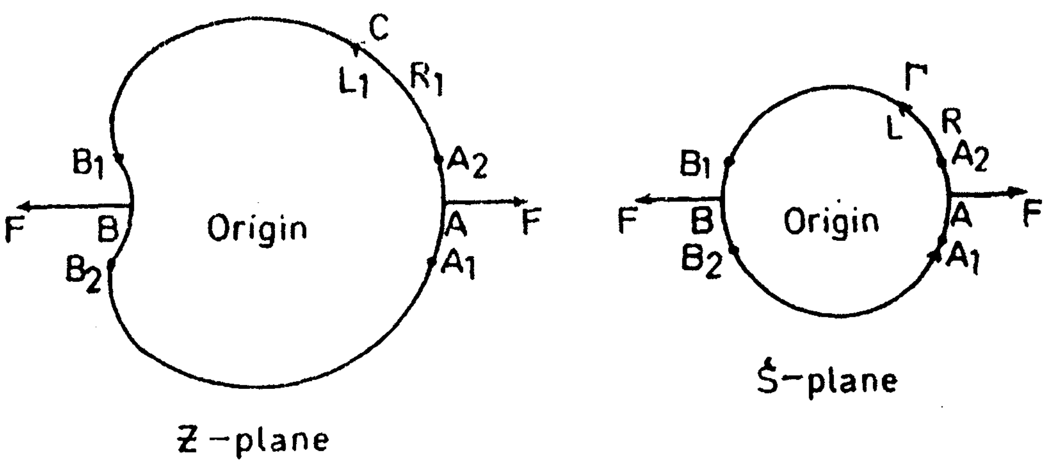

Fig.1 Transformation of Limacon into Circle.

The function y(ζ) has the form

$$\psi(\zeta) = A_0 + \frac{B_0}{\zeta} + \frac{c_0}{\zeta^2} \tag{2.7.3}$$

where A_0, B_0 and C_0 are unknown constants. These constants are determined by considering the analytic continuation of the complex potential function W(ζ) across circle Γ_1 and Γ_2 .

The second integral in equation (2.7.2) is identically zero as there is no force on Γ_2. Standard forces F at A and -F at B are obtained as the limit when $\in_1 \to 0$, $\in_2 \to 0$, of a uniform stress distribution:

$$-\rho(\sigma) + is(\sigma) = \begin{cases} \dfrac{F}{2a_1 \in_1} \; over\ arc\ A_1\ A\ A_2 \\ \dfrac{F}{2a_2 \in_2} \; over\ arc\ B_1\ B\ B_2 \end{cases} \tag{2.7.4}$$

Using the compatibility identity the unknown constants in(2.7.3), were determined arid than the stresses were found out for various values of α and β.The stress intensity factor $(\sigma_\eta)_\zeta=\beta$ for different values of β was discussed.

Some interesting results are given as follows :

(i) For each value of β, the stress intensity factor is maximum where the thickness is minimum.

(ii) As β →α, the eccentric annul us becomes crescet shaped and the stress intensity factor becomes large at the point of minimum thickness.

(iii) As $\beta \rightarrow 1^{+}$, the radius of the outer boundary of the eccentric becomes very large. In this case the stress-intensity factor becomes smaller as compared to its value when β is nearly equal to α.

Chapter-3

CHAPTER – III

A DIRECT METHOD OF COMPLEX VARIABLE

In this chapter the direct method developed by Sen [109] to solve fundamental plane problems in elasticity have been employed to find stresses in

(i) An infinite plate having, a hypotrochoidal hole under hydrostatic pressure,

(ii) An infinite plate with a parabolic crack under prescribed pressure.

In both the cases the infinite plate is assumed to be thin, isotropic and the stress and displacement components at great distances from the boundary of the hole (or the crack) are assumed to be zero.

3.1 INFINITE PLATE HAVING A HYPOTROCHOIDAL HOLE UNDER UNIFORM PRESSURE.

The mapping function is

$$z = R(\zeta + \frac{1}{3}\zeta^{-3}) \tag{3.1.1}$$

where $z = x+iy$, $\zeta = e^{\xi+i\eta}$, $R > 0$, $c \geq 0$

The boundary $\xi = 0$ gives a hypotrochoidal hole in the z-plane. In the absence of any body forces, the mean plane stress components σ_ξ, σ_η and $\tau_{\zeta\eta}$ satisfying the equations of equilibrium and compatibility are given by Sen [109] as follows, by Sen [109]:

$$\frac{8\sigma_\xi}{h^2} = \frac{\partial r^2}{\partial \eta} - \frac{\partial r^2}{\partial \xi}\frac{\partial \theta}{\partial \xi} + \frac{4\odot}{h^2} + F \qquad (3.1.2)$$

$$\frac{8\sigma_\eta}{h^2} = \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \eta} + \frac{4\odot}{h^2} - F \qquad (3.1.3)$$

$$\frac{8\tau_{\xi\eta}}{h^2} = -\frac{\partial r^2}{\partial \eta} - \frac{\partial \theta}{\partial \xi} - \frac{\partial r^2}{\partial \xi}\frac{\sigma\theta}{\partial \eta} - G \qquad (3.1.4)$$

Where $\frac{1}{h^2} = (\frac{\partial x}{\partial \xi})^2 + (\frac{\partial y}{\partial \xi})^2, r^2 = x^2 + y^2,$ $\qquad \theta = \sigma_\xi + \sigma_\eta$

and F and G are conjugate harmonic functions. The function $\odot$ is a plane harmonic function satisfying the equation

$$\frac{\partial^2\theta}{\partial \xi^2} + \frac{\partial^2\theta}{\partial \eta^2} = 0 \qquad (3.1.5)$$

Equations (3.1.2) to (3.1.4)can be rewritten for the convenience of use as under:

$$\frac{8\sigma_\xi}{h^4} = -\frac{\partial r^2}{\partial \eta}\frac{\partial}{\partial \eta}(\frac{\theta}{h^2}) - \frac{\partial r^2}{\partial \xi}\frac{\partial x}{\partial \xi}(\frac{\theta}{h^2}) + \frac{4\theta}{h^4} -$$

$$-\theta[\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}(\frac{1}{h^2})-\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}(\frac{1}{h^2})]+\frac{F}{h^2} \tag{3.1.6}$$

$$\frac{8\sigma_\eta}{h^4}=\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}(\frac{\theta}{h^2})-\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}(\frac{\theta}{h^2})+\frac{4\theta}{h^4}-$$

$$-\theta[\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}(\frac{1}{h^2})-\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}(\frac{1}{h^2})]-\frac{F}{h^2} \tag{3.1.7}$$

$$\frac{8\tau_{\xi\eta}}{h^4}=-\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\xi}(\frac{\theta}{h^2})-\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\eta}(\frac{\theta}{h^2})+$$

$$+\theta[\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\xi}(\frac{1}{h^2})+\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\eta}(\frac{1}{h^2})]-\frac{G}{h^2} \tag{3.1.8}$$

Boundary conditions

$$\sigma_\xi=-P,\ \tau_{\xi\eta}=0 \qquad at\ \xi=0 \tag{3.1.9}$$

Thus, we see that stresses σ_ξ, σ_η and $\tau_{\xi\eta}$ can be expressed explicitly in terms of functions F, G and θ. One can choose a suitable form of the harmonic functions θ involving certain constant coefficient, and hence determine an expression for G form (3.1.8). the relation between two conjugate function gives the expression for F, and then, from first boundary condition in 93.1.9) the assumed constant in θ can be determined from (3.1.6). Thus, F, G and θ being completely known, stresses can be found out.

Solution of the problem

From the mapping function (3.1.1), we get

$$\left.\begin{aligned} x &= R(e^{\xi}\cos\eta + \frac{1}{3}e^{-3\xi}\cos 3\eta) \\ y &= R(e^{\xi}\sin\eta - \frac{1}{3}e^{-3\xi}\sin 3\eta) \end{aligned}\right\} \qquad (3.1.10)$$

Hence,

$X^2=R^2+Y^2$

$$r^2 = r^2\left[e^{2\xi} + \frac{2}{3}e^{-2\xi}\cos 4\eta + + \frac{1}{9}e^{-6\xi}\right]$$

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2$$

$$\frac{1}{h^2} = R^2\{e^{2\xi} - 2e^{-2\xi}\cos 4\eta +^{-e6\xi} \qquad (3.1.11)$$

We assume θ to be given by, [55]:

$$\theta = B[1 - \mathrm{Re}\{(\zeta + \zeta^{-3})/(\zeta - \zeta^{-3})\}] \qquad (3.1.12)$$

Where B is an unknown constant and Re denotes the real part. The second condition of (3.1.9) with the help of (3.1.11) and (3.1.12) gives

$$G/h^2 = -\frac{64}{3}BR^4\sin 4\eta \qquad at \ \ \xi = 0 \qquad (3.1.13)$$

We notice that G is the imaginary part of the function

$$F + iG = \frac{16}{3}BR^4(\zeta + 3\zeta^{-3})/(\zeta - \zeta^{-3}) \qquad (3.1.14)$$

The function G obtained from (3.1.14) satisfies (3.1.13). Hence, we have,

$$G/h^2 = -\frac{64}{3}BR^4e^{-2\xi}\sin 4\eta \tag{3.1.15}$$

and

$$F/h^2 = \frac{16}{3}BR^4\{e^{2\xi} + 2e^{-2\xi}\cos 4\eta - 3e^{-6\xi}\} \tag{3.1.16}$$

The first condition of 3.1.9) then gives

$$B=-2P \tag{3.1.17}$$

The constant B being determined, F, G and θ are known and the problem is completely solved.

Expressions for stresses

Stress components σ_ξ, σ_η and $\tau_{\xi\eta}$ are determined by (3.1.6) & (3.1.8) in the following forms:

$$\frac{\sigma_\xi}{P} = \left\{3N^2 - HK - 2LT - \frac{2L(3N^2 + HM)}{T} - \frac{4}{3}M\right\}/T^2$$

(3.1.18)

$$\frac{\sigma_\eta}{P} = \left\{HK - 3N^2 - 2LT - 2L(3N^2 + HM)/t + \frac{4}{3}M\right\}/T^2$$

(3.1.19)

$$\frac{\tau_{\xi\eta}}{P} = N\{K + 3H + 2L(M - 3H)T - 4\}/T^2 \tag{3.1.20}$$

Where,

$$H = e^{2\xi} - \frac{2}{3}e^{-2\xi}\cos 4\eta - \frac{1}{3}e^{-6\xi}$$

$$K = 6e^{-6\xi} - 2e^{-2\xi}\cos 4\eta$$

$$L = e^{-6\xi} - e^{-2\xi}\cos 4\eta$$

$$M = e^{2\xi} + 2e^{-2\xi}\cos 4\eta - 3e^{-6\xi} \tag{3.1.21}$$

$$N = \frac{4}{3}e^{-2\xi}\sin 4\eta$$

$$T = e^{2\xi} - 2e^{-2\xi}\cos 4\eta + e^{-6\xi}$$

The stress-intensity factor is given by

$$[\sigma_\eta]_{\xi=0} \tag{3.2.22}$$

Particular cases

(i) Circular hole

When c=0, the hole is a circle. The stress-intensity factor is given by

$$S(m, 0) = P \tag{3.1.23}$$

As a check the solution, stress components σ_ξ and $\tau_{\xi\eta}$ have been calculated from equations (3.1.18) and (3.1.20) are found to satisfy the boundary conditions (3.1.9).

(ii) Approximate square

The boundary $\xi = 0$ becomes an approximate square with rounded corners. (Fig. 2).

3.2 INTEGRAL EQUATIONS AND INTEGRAL TECHNIQUE

We have seen that the problems of plates with a curvilinear boundary under certain load condition or those of infinite plates having a hole with a smooth curvilinear boundary can be conveniently solved by mapping the curvilinear boundary on a circle by some suitable conformal mapping function. The problems of determining stresses in the neighbourhood of a crack in a plate under certain system of stress can be very elegantly solved .by integral transforms. I.N. Sneddon [119] made a very remarkable contribution in this direction.

A crack in a three dimensional body can be considered as a plane curve if the body is in a state of plane stress or strain. However, if this curve is a straight line segment, it is known as a Griffith crack. In realty a Griffith crack is a flat ribbon-shaped cavity in the solid. The problems can be solved as a plane strain case, since the plane stress can then be deduced simply by changing the value of poisson's ratio of the material. For simplicity the elastic body is supposed to be homogeneous and isotropic. If has been found that cracks exist or develop in a

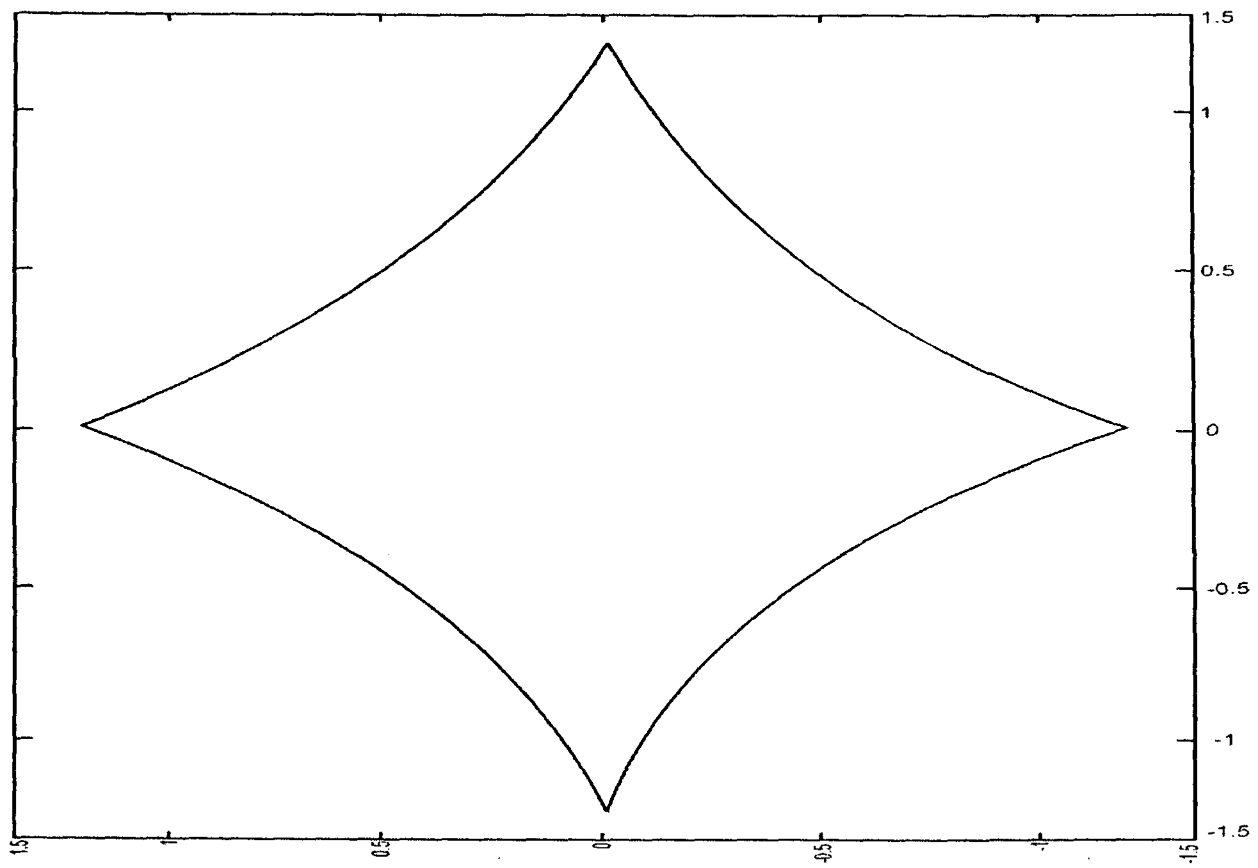

FIG. 2 HYPOTROCHOIDS

$$z = R\left(\zeta + \frac{1}{3}\zeta^{-3}\right)$$

plate when it is, subjected to certain mode of stress beyond certain limit. There are three modes of displacements.

(i) In mode I displacement, the plate is subjected to tensile forces at its boundary.

(ii) In mode II the displacement solid is under an applied shear parallel to crack.

(iii) In mode III the displacement solid is under an applied shear perpendicular to the crack.

Inglis [52] discussed the distribution of stresses in the neighbourhood of an elliptic crack in a thin plate. Later on Griffith [44], considered the limiting case of the results obtained by Inglis in which the semi-minor axis of the ellipse was zero. Thus the ellipse degenerated to a straight line segment of length 2c, which was termed as 'Griffith crack' considering the infinite plate to occupy the xy-plane, the crack may be represented as

$$y = o, -c \leq x \leq c \qquad (3.2.1)$$

Generally, we discussed two types of problems. The first one consists of those in which the internal pressure opening the crack varies along the length of the crack. The governing equations are

$$\sigma_y = -\rho(x), \tau_{xy} = 0, \quad \text{on } -c \leq x \leq c, \ y = 0 \qquad (3\ 2.2)$$

and

$$\tau_{xy} = 0 \quad \sigma_x = 0, \quad \sigma_y = 0, \ as \ \sqrt{x^2 + y^2} \to \alpha \qquad (3.2.3)$$

where $\rho(x)$ is a prescribed function of x. If we suppose that the field is symmetrical about the x-axis. As a result of this symmetry, outside the crack, the component of stress is given by the prescribed component of the displacement vector normal to the x-axis and the shearing stress τ_{xy} are both zero. Inside the crack the normal component of stress is given by the prescribed component of stress is given by the prescribed function p(x) and the shearing stress is zero.

$$\sigma_y = -\rho(x),\ \tau_{xy}=0 \quad |x|\leq c,\ y = 0 \qquad (3.2.4)$$

$$\tau_{xy} = 0,\ U_y = 0, = \quad |x|\geq c,\ y = 0 \qquad (3.2.5)$$

and σ_x, σ_y and τ_{xy} all tend to zero at infinity. If in addition, $\rho(x)$ is supposed to be an even function of x, we have the equations

$$\sigma_y = \rho(x),\ \tau_{xy} = 0, \quad 0\leq c,\ y = 0 \qquad (3.2.6)$$

$$\tau_{xy} = 0,\ U_y = 0, \quad x\geq c,\ y = 0 \qquad (3.2.7)$$

In the second type of problems the crack is opened in a prescribed shape. In these problems the distribution of pressure necessary to produce *a* Griffith crack of prescribed shape is to be found out. I.N. Snedden [119] used the following boundary conditions applied to half plane y =o

$$\left.\begin{array}{lll} U_y & = & w(x)H(1-|x|), \qquad w(1)=0, \; on\; y=0 \\ \tau_{xy} & = & 0, \; on\; y=0 \end{array}\right] \qquad (3.2.8)$$

to determined the value of σ_y (x,o),where the function H(x-t) is given by

$$\int_0^\infty J_0(\zeta t)\sin(\zeta x)d\zeta = \frac{H(x-t)}{\sqrt{x^2-t^2}}$$

J_0 being the Bessel's function of order zero. The Fourier cosine transform has been employed to solve the problem. Taken some simple forms of w(x) various results have been obtained. Recently Ali has discussed a similar problem but the body was assumed to be initially stressed. These; problems are much more common in realty. Bodies are found to be in a state of initial stress in equilibrium condition. If a length of wire be taken and its end welded, the circular hoop will be in a state of initially stress and the unstressed state can not, be obtained without cutting the hoop open. These initial stresses generally cause finite deformation. Now, if this initially stressed body is further subjected to infinitesimal deformation, the state of body can be studied as the superposition of later on the former by the method given by A.E. Green et. al. [52].The fundamental equation of incremental deformation theory constructed by Biot [17] and Kurashige [61] was adopted. The body was assumed to be isotropic homogeneous and in a state of plane strain. The effect of the initial finite deformation was studied on the distribution of pressure necessary to produce a two-dimensional Griffith crack of prescribed shape in an infinite body.

The initial finite deformation is assumed is to produce only a normal stress component which is perpendicular to the crack plane y=o. The Fourier transform is given by

$$\left.\begin{aligned} \bar{\phi} &= \int_{-\infty}^{\infty} \phi e^{i\zeta x} dx \\ \phi &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\phi}\bar{e}^{i\zeta x} d\zeta \end{aligned}\right] \qquad (3.2.9)$$

where $\bar{\phi}$ is the Fourier transform of ϕ. It is supposed that the prescribed shape was given by

$$U_y\,(x,0)=w(x),\ |x| \leq c$$

under the action of the pressure, p(x) on y = 0. The boundary conditions are:

$$\left.\begin{array}{llcl} (i) & \tau_{xy}(x,0) & = & 0 \ \textit{for all}\ x \\ (ii) & \sigma_y(x,0) & = & -p(x),\ \textit{for all}\ x \\ (iii) & U_y(x,0) & = & w(x), |x| \leq c \\ (iv) & U_y(x,0) & = & 0, |x| > c \end{array}\right] \qquad (3.2.10)$$

The problem was ultimately reduced to a pair of dual integral equations

$$\left.\begin{aligned} &\frac{2}{\pi}\int_0^\infty \frac{\overline{\rho(\zeta)}}{\zeta}\cos\zeta x\, d\zeta = G w(x),\ x \le c \\ &\int_0^\infty \frac{\overline{\rho(\zeta)}}{\zeta}\cos\zeta x\, d\zeta = 0,\ x > c \end{aligned}\right] \tag{3.2.11}$$

Solving the above integral equations $\overline{\rho(\zeta)}$ and hence $\bar{\phi}$ was determined. By taking

$$w(x) = \in \left(1 - \frac{x^2}{c^2}\right) \tag{3.2.12}$$

where $\in$ is a small positive number, the crack becomes parabolic. The normal stress σ_y (x,o) was determined. The variation of the pressure $\rho(x)$ was studied for different values of x. The elliptic shape of the crack was given by

$$w(x) = \in \left(1 - \frac{x^2}{c^2}\right)^{\frac{1}{2}} \quad 0 \le x \le c \tag{3.2.13}$$

The Fourier cosine transform has been used to solve the problem and the normal stress $\sigma_y(x,o)$ on the crack was determined .It was found that the pressure necessary to produce a Griffith crack of elliptic shape in an initially stressed solid should be uniform. This was in agreement with the result obtained by Sneddon [119].

Later on a mixed value problem of a pair of Griffith crack in an infinite solid has been considered [7]. The body is supposed to be homogeneous and isotropic material and in a state of plane strain. The two cracks are supposed to occupy $-b \leq y \leq -a$ $(a<b)$ and $a \leq y \leq b$ on x=o. Thus each of the cracks has a length b-a. The boundary conditions are

$$\left.\begin{array}{ll} \sigma_x(0,y) = -\rho(y), & (a \leq |y| \leq b) \\ \tau_{xy}(0,y) = 0 & (-\infty < y < \infty) \\ U(0,y) = 0 & (|y| < a,\ |y| > b) \end{array}\right] \tag{3.2.14}$$

The function ρ(y) has been assumed to be even so that there is a symmetry about the plane y=0. Using the Fourier transform technique the components of stress and strain are given by Sneddon [119] as follows

$$\left.\begin{array}{ll} \sigma_x = & -\dfrac{2}{\pi}\displaystyle\int_0^\infty \overline{\rho(\zeta)}(1+\zeta x)e^{-\zeta x}\cos\zeta x\, d\zeta \\ \sigma_y = & -\dfrac{2}{\pi}\displaystyle\int_0^\infty \overline{\rho(\zeta)}(1-\zeta x)e^{-\zeta x}\cos\zeta x\, d\zeta \\ \tau_{xy} = & -\dfrac{2}{\pi}\displaystyle\int_0^\infty \bar{\rho}(\zeta)e^{-\zeta x}\sin\zeta y\, d\zeta \end{array}\right] \tag{3.2.15}$$

and

$$u = \frac{2(1+\mu)}{E_0}\int_0^\infty \bar{\rho}(\zeta)e^{-\zeta x}\, 2[1-\mu-\zeta x]^{1/\zeta}\cos\zeta y\, d\zeta$$

$$v = \frac{-2(1+\mu)}{E_0}\int_0^\infty \bar{\rho}(\zeta)e^{-\zeta x}\, 2[1-\mu\zeta x]^{1/\zeta}\sin\zeta y\, d\zeta$$

Where $\bar{\rho}(\zeta)$, and even function of ζ is the Fourier cosine transform of ρ(y). We notice that the second boundary condition of (3.1.14) is automatically satisfied and the other two boundary conditions given rise to the triple integral equations

$$\left.\begin{aligned} &\int_0^\infty \bar{\rho}(\zeta)\cos\zeta y \, d\zeta = 0 && (0 < y < a) \\ &\int_0^\infty \bar{\rho}(\zeta)\cos\zeta y \, d\zeta = \frac{\pi}{2}p(y) && (0 < y < a) \\ &\int_0^\infty \frac{1}{\zeta}\bar{\rho}(\zeta)\cos\zeta y \, d\zeta = 0 && (y > b) \end{aligned}\right] \qquad (3.2.17)$$

Following the method of Srivastava and Lowengrub [69] the solution of equations (3.2.17) has been obtained. For the special case

$$p(y) = \rho \qquad \text{(a constant)} \qquad (3.2.18)$$

The solution has been obtained and stresses on the line of crack are determined in terms of elliptic integrals. The normal stress $\sigma_x(0,y)$ has been studied graphically for different values of a and b. It was observed that the nature of the normal stress in two regions $|y|<a$ and $|y|>b$ were of opposite nature. It was also observed that the normal stress at a point bisecting the line joining the centres of pair of cracks each of unit length, decreases as the distance between centres of the cracks increases where as it increases as the length of each cracks increases.

If a=o, the two cracks combine to give a single Griffith crack of length 2b and the results so obtained agreed .apart from a trivial change of notations, with those obtained by Sneddon and Elliot[118].

Similarly, the problem of a body having several cracks along a line can be considered Green and Zerna [42] discussed similar problems. It was supposed that an infinite two-dimensional medium contained n collinear cracks situated along the segments.

$$L_r = a_r \; b_r \quad (r=1, \ldots, n) \tag{3.2.19}$$

of the real axis and that all components of stress vanish of infinity. The cracks are supposed to be opened symmetrical of about the x-axis by a given normal distribution of displacement along the line segments $L = L_1 + L_2 + \text{---} + L_n$ in the first type of problems. In the second type of problems the cracks are opened symmetrically by a given distribution of normal stress along the line segment L.

Thus the boundary conditions are

Problem 1:

$$\left.\begin{aligned} U_y &= f(x) \; on < \\ \tau_{xy} &= 0, \; everywhere \; on \; ox \\ U_y &= 0, \; on \; ox \; outside \; L \end{aligned}\right] \tag{3.2.20}$$

Problem 2:

$$\left.\begin{aligned} U_y &= \rho(x) \text{ on} < \\ \tau_{xy} &= 0, \textit{ everywhere on ox} \\ U_y &= 0, \textit{ on ox outside } L \end{aligned}\right] \qquad (3.2.21)$$

where f(x) and p(x) are piecewise continuous function on L. Employing the above boundary conditions in the governing equations we can solve the problems of determining stresses in the neighbourhood of prescribed shape or cracks opened by given pressure.

Chapter-4

CHAPTER – IV

INTEGRAL TRANSFORM TECHNIQUE

The technique of integral transform has been found to be very useful in the case of crack problems. The major contribution towards the development of the subject in this direction is due to Sneddon [117] and co-workers. I.N. sneddon and M. Lowengrub [119] have written a monograph on crack problems in the classical theory of elasticity. Crack and punch problems for transversely isotropic bodies have been solved by Eiliot [26], using Hankel's transform. Sneddon and Eilliot [118] have solved the distribution of stress in the neighbourhood of a crack which is subject to an internal pressure varying along the length of the crack. Using integral transform technique, Ali, M. [9] has discussed the case of opening of a crack of prescribed shape in an initially stressed body. Later on, Parray [100] solved a punch problem for an initially stressed Neo-Hookean solid. In solving the boundary value problems in elasticity the following transforms are very frequently used:

I. Fourier Transform:

The Fourier transform is defined as

$$F(\rho) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(\xi) e^{i\rho\xi} d\xi,$$

With its inverse transform

$$f(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty F(\rho) e^{-i\rho\xi} d\rho.$$

The Fourier transform is used for functions whose domain is the whole real line. If the domain is the positive real line the Fourier cosine, sine transforms defined as given below are used:

$$F_c(\rho) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi)\cos(\rho\xi) d\xi,$$

$$F_s(\rho) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi)\sin(\rho\xi) d\xi,$$

with the inversion formulas given respectively as :

$$f(\zeta) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\rho)\cos(\rho\xi) d\rho,$$

$$f(\zeta) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\rho)\sin(\rho\xi) d\rho.$$

II. Hankel transform :

Hankel transform of order r is defined in terms of $J_r(\rho\zeta)$, the

Bessel function of the first kind of order r, as follows :

$$H(\zeta)=\int_0^{\alpha} \rho f(\rho)J_r(\rho\xi)d\rho,$$

with its inversion formula given as :

$$f(\rho)=\int_0^{\alpha} \zeta H(\zeta)J_r(\rho\xi)d\zeta,$$

III. <u>Mellin transform :</u>

Mellin transform of a function f(ζ) whose domain is the positive real line is defined as follows :

$$F(\zeta)=M\ [f(\rho)\zeta]\ =\int_0^{\alpha} \rho^{\xi-1}f(\rho)d\rho,$$

with the inversion formula given by :

$$f(\rho)=M^{-1}\ [F(\zeta);\rho]\ =\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma-i\infty} \rho^{-\xi}F(\zeta)d\zeta,$$

The integral on the right side taken over a line parallel to the imaginary axes at a distance γ, in the ζ-plane and to the right of all the singularities of f(ζ).

In the foregoing section, a three-dimensional mixed boundary value problem for an initially stressed body has been discussed.

4.1 SMALL DEFORMATION OF AN INITIALLY STRESSED BODY:

Various elastic bodies are found to possess some initial stress which exists in the body by process of preparation or by the action of body forces. For example, if a sheet of metal is rolled up into a cylinder and edges welded together, the body so formed is in a state of initial stress and the unstrained state cannot be attained without cutting the cylinder open.

These initial stresses may be assumed to cause finite deformation. If in addition to this, the body is subjected to further small deformation, the state of the body can be studied as the superposition of the latter on the former [41]. Kurashige [61], discussed a two-dimensional crack problem for an initially stressed neo-Hookean solid and later on discussed a circular crack problem for the same.

In this chapter attempts have been made to (a) obtain a basic equation to be satisfied by a displacement function and to express the components of incremental displacement, stress and strain by it for an axisymmetric problem under incremental deformation theory of neo-Hookean solid and (b) obtain the solution of a crack problem for initially stressed body.

Basic Equations :

We have adopted the fundamental equation of incremental deformation theory constructed by Biot [12] and Kurashige [61]. In rectangular cartesian coordinates x_i and t. the equations of motion associated with incremental deformation theory of elasticity are :

$$\frac{\partial s_{ij}}{\partial x_j} + S_{jk}\frac{\partial w_{ik}}{\partial x_j} + S_{ik}\frac{\partial w_{jk}}{\partial x_j} - e_{jk}\frac{\partial S_{ik}}{\partial x_j} = Q\frac{\partial^2 u_i}{\partial t^2}, \qquad (4.1.1)$$

And

$$\Delta f_i = \left(S_{ij} + S_{kj} w_{ik}\ e - S_{ik} e_{jk}\right) n_j \qquad (4.1.2)$$

where

x_1 = Cartesian coordinates,

S_{ij} = initial stress, corresponding to initial finite deformation, referred to x_1

u_i = incremental displacement (Infinitesimal),

e_{ij} = incremental strain,

w_{ij} = incremental rotation,

e = incremental volume expansion,

s_{ij} = incremental stress referred to axis which are incrementally displaced with the medium,

Δf_i = incremental boundary force per unit initial area,

n_i = component of unit normal to boundary surface,

Q = density in a finite deformation.

In equations (4.1.1) and (4.1.2), the usual convention for summation over repeated indices is applied The second, third and fourth terms of left-hand side in equation (4.1.1) represent the effect of initial stress.

The incremental strain-displacement relations can be written in the same forms as in classical elasticity, because the incremental displacement is infinitesimal. Hence we have:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\right) \tag{4.13}$$

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\right) \tag{4.1.4}$$

Taking the material to be a so-called neo-Hookean solid, elastic potential per unit volume is expressed in the form [132]:

$$w = \frac{1}{2}\mu_0\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right), \tag{4.1.5}$$

with

$$\lambda_1\ \lambda_2\ \lambda_3 = 1 \tag{4.1.6}$$

where λ_1 is extension ratios and μ_0 is shear modulus in an unstrained state.

The initial stresses satisfy the following stress-strain relations:

$$S_{11} - S_{22} = \mu_0\ (\lambda_1^2 - \lambda_2^2), \tag{4.1.7a}$$

$$S_{22} - S = \mu_0\ (\lambda_1^2 - \lambda_3^2), \tag{4.1.7b}$$

$$S_{33} - S_{11} = \mu_0\ (\lambda_3^2 - \lambda_1^2), \tag{4.1.7c}$$

The total differentiation of equation (4.1.7) and consideration of incremental shear deformation give the following incremental stress-stream relations:

$$S_{11} - S_{22} = 2\mu_0 (\lambda_1^2 e_{11} - \lambda_2^2) e_{22}, \tag{4.1.8a}$$

$$S_{22} - S_{33} = 2\mu_0 (\lambda_2^2 e_{22} - \lambda_3^2) e_{33}, \tag{4.1.8b}$$

$$S_{33} - S_{11} = 2\mu_0 (\lambda_3^2 e_{33} - \lambda_1^2) e_{11}, \tag{4.1.8c}$$

and

$$S_{12} = \mu_0 (\lambda_1^2 + \lambda_2^2) e_{12} \tag{4.1.9a}$$

$$S_{23} = \mu_0 (\lambda_2^2 + \lambda_3^2) e_{23} \tag{4.1.9b}$$

$$S_{31} = \mu_0 (\lambda_3^2 + \lambda_1^2) e_{31} \tag{4.1.9c}$$

4.2 AXISYMMETRIC INCREMENTAL DEFORMATION:

The half plane is assumed to be deformed under the impact of an axisymmetric body. Let (ρ,θ, z) be the cylindrical polar coordinates of a point in the initially deformed body, and they are connected with rectangular coordinates by

$$\rho = \sqrt{x_1^2 + x_2^2}, \tag{4.2.1a}$$

$$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right), \tag{4.2.1b}$$

$$z = x_3 \tag{4.2.1c}$$

It is assumed that non-zero components of initial stress are $s_{\rho\rho}$, $s_{\theta\theta}$ and S_{zz} which are uniform throughout a body, and that the body is in the state of syminetrical incremental strain with respect to the z-axis. The equations of motion (4.1.1) reduce in the cylindrical polar coordinates to :

$$\frac{s_{\rho\rho}}{\partial\rho}+\frac{s_{\rho\rho}-s_{\theta\theta}}{\rho}+\frac{\partial s_{\rho z}}{\partial z}-(S_{\rho\rho}-S_{zz})\frac{\partial w_{\rho z}}{\partial z}=Q\frac{\partial^2 u_\rho}{\partial t^2} \tag{4.2.2}$$

$$\frac{\partial s_{zz}}{\partial z}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\partial s_{\rho\rho}\right)-(S_{\rho\rho}-S_{zz})\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho w_{\rho z})=Q\frac{\partial^2 u_z}{\partial t^2} \tag{4.2.3}$$

where the last terms in the left-hand side represent the effect of initial stress and, if the initial stress is small, are negligible in comparison with other terms. The incremental displacements u_ρ and u_z in terms of potential function $\phi(\rho,z)$ are given as follows:

$$u_\rho=-\frac{\partial^2\phi}{\partial\rho\partial z} \tag{4.2.4a}$$

$$u_z=\frac{1}{\rho}\frac{\rho}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right), \tag{4.2.4b}$$

where ϕ is a scalar function of p and z. Further it is easily checked that the condition for incompressibility

$$e=e_{\rho\rho}+e_{\theta\theta}+e_{zz}=0 \tag{4.2.5}$$

is satisfied. The non-zero components of incremental strain, rotation and stress can be expressed in terms of the function ϕ as follows:

$$e_{\rho\rho}=-\frac{\partial^3\phi}{\partial\rho^2\partial z},\ e_{\theta\theta}=-\frac{1}{\rho}\frac{\partial^2\phi}{\partial\rho\partial z},\ e_{zz}=\frac{1}{2}\frac{\rho}{\partial\rho}\left(\rho\frac{\partial^2\phi}{\partial\rho\partial z}\right),$$

$$e_{z\rho}=\frac{1}{2}\frac{\partial}{\partial\rho}\left\{\frac{1}{\rho}\frac{\rho}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho\partial z}\right)-\frac{\partial^2\phi}{\partial z^2}\right\}, \tag{4.2.6a}$$

$$w_{\rho z} = -\frac{1}{2}\frac{\partial}{\partial\rho}\left\{\frac{1}{\rho}\frac{\rho}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)+\frac{\partial^2\phi}{\partial z^2}\right\}, \tag{4.2.6b}$$

$$s_{\rho\rho} - s = -\mu_0\lambda_\rho^2\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial^2\phi}{\partial\rho\partial z}\right), \tag{4.2.7a}$$

$$s_{\theta\theta} - s = \mu_0\lambda_\rho^2\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial^2\phi}{\partial\rho\partial z}\right), \tag{4.2.7b}$$

$$s_{zz} - s = \mu_0\left(\lambda_\rho^2 + 2\lambda_z^2\right)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial^2\phi}{\partial\rho\partial z}\right), \tag{4.2.7c}$$

$$S_{z\rho} = \frac{1}{2}\mu_0\left(\lambda_\rho^2 + \lambda_z^2\right)\frac{1}{\partial\rho}\left\{\frac{1}{\rho}\frac{\rho}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)-\frac{\partial^2\phi}{\partial z^2}\right\}, \tag{4.2.7d}$$

where

$$s = \frac{1}{2}(s_{\rho\rho} + s_{\theta\theta}) \tag{4.2.8}$$

Substitution from expressions (4.2.6) and (4.2.7) into equations (4.2.2) and (4.2.3) give the following equation

$$\frac{\partial s}{\partial\rho} - \frac{1}{2}\frac{\partial s}{\partial\rho}\left[\begin{array}{l}\left\{\mu_0\left(\lambda_\rho^2+\lambda_z^2\right)+\left(S_{\rho\rho}-S_{zz}\right)\right\}\frac{1}{\rho}\frac{\partial s}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)+\\ +\left\{\mu_0\left(\lambda_\rho^2+\lambda_z^2\right)-\left(S_{\rho\rho}-S_{zz}\right)\right\}\frac{\partial^2\phi}{\partial z^2}\end{array}\right]+$$
$$+\left\{\mu_0\lambda_z^2+\left(S_{\rho\rho}-S_{zz}\right)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial^2\phi}{\partial\rho\partial z}\right)\right\} = -Q\frac{\partial^3\phi}{\partial z\partial t^2} \tag{4.2.9a}$$

$$\frac{\partial s}{\partial \rho}+\frac{1}{2}\frac{1}{\rho}\left(\rho\frac{\partial s}{\partial \rho}\right)\left[\begin{array}{l}\left\{\mu_0\left(\lambda_\rho^2+\lambda_z^2\right)+\left(S_{\rho\rho}-S_{zz}\right)\right\}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \phi}{\partial \rho}\right)+\\ +\left\{\mu_0\left(\lambda_\rho^2+\lambda_z^2\right)-\left(S_{\rho\rho}-S_{zz}\right)\right\}\frac{\partial^2 \phi}{\partial z^2}\end{array}\right]+ \tag{4.3.9b}$$

$$+\left\{\mu_0\lambda_z^2+\left(S_{\rho\rho}-S_{zz}\right)\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial^2 \phi}{\partial \rho\partial z}\right)\right\}=Q\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho\frac{\partial^3 \phi}{\partial \rho\partial t^2}\right]$$

And further substitution from the first of equations (4.2.9) into the second and use of stress-strain relations

$$S_{\rho\rho}-S_{\theta\theta}=\mu_0\left(\lambda_\rho^2-\lambda_\theta^2\right) \quad = \quad 0, \tag{4.2.10a}$$

$$S_{\rho\rho}-S_{zz}=\mu_0\left(\lambda_\rho^2-\lambda_\theta^2\right) \quad = \quad 0, \tag{4.2.10b}$$

Which are reduced from equations (4.1.7) give a simple partial differential equation of ϕ and expression for stresses as follows:

$$\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right)+\frac{\partial^2}{\partial z^2}\right\}\left\{k^2\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \phi}{\partial \rho}\right)+\frac{\partial^2 \phi}{\partial z^2}-\frac{Q}{\mu_0}\frac{\partial^2 \phi}{\partial t^2}\right\}=0 \tag{4.2.11}$$

$$s = \mu_0\lambda_z^2\frac{\partial^3 \phi}{\partial z^3}-Q\frac{\partial^3 \phi}{\partial z\partial t^2}, \tag{4.2.12}$$

where

$$k = \lambda\rho/\lambda z \tag{4.2.13}$$

4.3 CRACK PROBLEM FOR AN INITIALLY STRESSED NEO-HOOKEAN SOLID : FORMULATION OF PROBLEM

The semi-infinite medium $z \geq 0$ is supposed to be initially deformed and components S_{zz} , in addition to $S_{\theta\theta}$ is also supposed to be zero so that

$$S\rho\rho = \mu_0(\lambda_\rho^2 \; - \lambda^2{}_z) = -P \tag{4.3.1}$$

Further, without loss of generality the length of crack is taken to be unity in the plane z = 0, the centre of crack lying on the z-axis The distribution of stress in the neighbourhood of the crack is studied with the help of Hankel transform.

The Hankel transformation and its inversion transform of order zero are given by following relations:

$$\phi(\xi) = \int_0^\infty \rho\phi(\rho)J_0(\rho\xi)d\rho, \tag{4.3.2a}$$

$$\phi(\rho) = \int_0^\infty \xi\bar{\phi}(\xi)J_0(\rho\xi)d\xi, \tag{4.3.2b}$$

Which of the following formulae for differentiated function are obtained;

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) = -\int_0^\infty \bar{\phi}\xi^3 J_0(\rho\xi)d\xi, \tag{4.3.3a}$$

$$\frac{\partial}{\partial\rho}\phi = \int_0^\infty \bar{\phi}\xi^2 J_1(\rho\xi)d\xi, \tag{4.3.3b}$$

Using these formulae, the incremental displacements u_ρ and u_z, the incremental rotation $w_{\rho z}$ and the components of incremental stress s_{zz}, $s_{z\rho}$, S are expressed in Hankel inversion as follows:

$$u_\rho \quad = \quad \int_0^\infty \frac{\partial\bar{\phi}}{\partial z}\zeta^2 J_1(\rho\zeta)d\zeta \tag{4.3.4a}$$

$$u_z \quad = \quad -\int_0^\infty \bar{\phi}\zeta^3(\rho\zeta)d\zeta \qquad (4.3.4b)$$

$$w_{\rho z} \quad = \quad -\frac{1}{2}\int_0^\infty \left(\zeta^4\bar{\phi}-\zeta^2\frac{\partial^2\bar{\phi}}{\partial z^2}\right)J_1(\rho\zeta)d\zeta \qquad (4.3.5)$$

$$s_{zz} \quad = \quad s-\mu_0\lambda_z^2(2+k^2)\int_0^\infty \zeta^3\frac{\partial\bar{\phi}}{\partial z}J_0(\rho\zeta)d\zeta \qquad (4.3.6a)$$

$$s_{z\rho} \quad = \quad \frac{1}{2}\mu_0\lambda_z^2(1+k^2)\int_0^\infty \zeta^4\bar{\phi}+\zeta^2\frac{\partial^2\bar{\phi}}{\partial z^2}J_1(\rho\zeta)d\zeta, \qquad (4.3.6b)$$

$$s \quad = \quad \mu_0\lambda_z^2\int_0^\infty \zeta\frac{\partial^3\bar{\phi}}{\partial z^3}J_0(\rho\zeta)d\zeta \qquad (4.3.c6)$$

Writing the first of equations (4.2.2) and (4.2.3) in the form

$$\frac{\partial}{\partial\rho}(\rho^2 s_{\rho\rho}) \quad = \quad \rho^2\left\{\frac{s_{z\rho}+s_{\theta\theta}}{\rho}-\frac{\partial s_{z\rho}}{\partial z}-\mu_0\lambda_z^2(1-k^2)\frac{\partial w_{\rho z}}{\partial z}\right\} \qquad (4.3.7)$$

Substituting from expressions (4.3.5) and (4.3.6) into the above equation and integrating it with respect to ρ lead to the following expression for $s_{\rho\rho}$:

$$s_{\rho\rho} = \mu_0\lambda_z^2\left\{\begin{array}{c}\int_0^\alpha \left(k^2\zeta^3\frac{\partial\bar{\phi}}{\partial z}+\zeta\frac{\partial^3\bar{\phi}}{\partial z^3}\right)J_0(\rho\zeta)d\zeta - \\ -k^2\frac{2}{\rho}\int_0^i \zeta^2\frac{\partial\bar{\phi}}{\partial z}J_i(\rho\zeta)d\zeta\end{array}\right\} \qquad (4.3.8)$$

By applying Hankel transform the basic equation (4.2.11) reduces to the following ordinary differential equation:

$$\left(\frac{\partial^2}{\partial z^2}-\zeta^2\right)\left(\frac{\partial^2\bar{\phi}}{\partial z^2}-k^2\zeta^2\bar{\phi}\right)=0 \qquad (4.3.9)$$

Boundary Conditions:

If the pressure P(ρ) is supposed to be applied to the plane z=0, the boundary conditions for the semi infinite body are presented as follows:

$$s_{\rho z} = 0 \qquad (0 \le \rho \le \infty) \qquad (4.3.10a)$$

$$s_{zz} = -p(\rho) \qquad (0 \le \rho \le \infty) \qquad (4.3.10b)$$

$$u_z = 0 \qquad (1 < \rho < \infty) \qquad (4.3.11a)$$

$$s_{zz} = -\rho \qquad (0 \le \rho \le \infty) \qquad (4.3.11b)$$

Solution of the Problem:

The solution of the differential equation (4.3.9) for half-space z $\geq$ 0 is given by

$$\bar{\phi} = A(\zeta)e^{\zeta z} + B(\zeta)e^{k\zeta z}, \qquad (4.3.12)$$

Where A(ζ) and B(ζ) are integral constants. Application of the boundary conditions (4.3.10) to equation (4.3.12) gives

$$A(\zeta) = \left[\frac{(1+k^2)}{\mu_0\lambda_z^2\{(1+k^2)^2-4k\}}\right]\frac{\overline{p(\zeta)}}{\zeta^3}, \qquad (4.3.13a)$$

$$B(\zeta) = \left[\frac{-2}{\mu_0\lambda_z^2\{(1+k^2)^2-4k\}}\right]\frac{\overline{p(\zeta)}}{\zeta^3}, \qquad (4.3.13b)$$

The boundary conditions (4.3.11) applied to equation (4.3.12) give the following dual integral equations

$$\int_0^\infty \zeta\,\overline{p(\zeta)}J_0(\rho\zeta)\,d\zeta - \rho \qquad (\rho<1), \qquad (4.3.14a)$$

$$\int_0^\infty \overline{p(\zeta)}J_0(\rho\zeta)\,d\zeta - \rho \qquad (\rho>1), \qquad (4.3.14b)$$

where $\overline{p(\zeta)}$ is an unknown function. The above dual integral equations are a special case of the pair of equations considered by E.G. Titchmarsh [131] and therefore, the solution of equations (4.3.14) are given as follows :

$$\overline{p(\zeta)} = \frac{1}{2}\left\{\frac{2\cos\zeta}{\zeta^3}+\frac{2\sin\zeta}{\zeta^2}-\frac{\cos\zeta}{\zeta}-\frac{2}{\zeta^3}\right\} \tag{4.3.15}$$

Thus, $\overline{p(\zeta)}$ being determined, the constants A(ζ), B(ζ) and $\overline{p(\zeta)}$ are known. Now the function ϕ can easily be determined from the equation (4.3.12). Hence the problem is completely solved.

The non-vanishing components of incremental displacement and incremental stress are found in terms of the Hankel inversion as follows:

$$\begin{aligned} u_z &= -\frac{1}{\mu_0\lambda_z^2\{(1+k^2)^2-4k\}}\int_0^\infty \left\{(1+K^2)e^{\zeta z}-2e^{k\zeta z}\right\}x \\ &\quad x\left\{\frac{\cos\zeta}{\zeta^3}+\frac{\sin\zeta}{\zeta^2}-\frac{\cos\zeta}{2\zeta}-\frac{1}{\zeta^3}\right\}J_0(\rho\zeta)d\zeta, \end{aligned} \tag{4.3.16a}$$

$$\begin{aligned} u_\rho &= \frac{1}{\mu_0\lambda_z^2\{(1+k^2)^2-4k\}}\int_0^\infty \left\{(1+K^2)e^{\zeta z}-2ke^{k\zeta z}\right\}x \\ &\quad x\left\{\frac{\cos\zeta}{\zeta^3}+\frac{\sin\zeta}{\zeta^2}-\frac{\cos\zeta}{2\zeta}-\frac{1}{\zeta^3}\right\}J_0(\rho\zeta)d\zeta, \end{aligned} \tag{4.3.16b}$$

$$s_{zz} = \frac{1}{\{(1+k^2)^2-4k\}}\int_0^\infty \left\{(1+K^2)^2 e^{\zeta z} - 4ke^{k\zeta z}\right\} x$$
$$x\left\{\frac{\cos\xi}{\zeta^2}+\frac{\sin\zeta}{\zeta}-\frac{\cos\zeta}{2}-\frac{1}{\zeta^2}\right\}J_0(\rho\zeta)d\zeta \qquad (4.3.17a)$$

$$s_{z\rho} = -\frac{1}{\{(1+k^2)^2-4k\}}\int_0^\infty \left\{(1+k^2)^2 (e^{\zeta z} - e^{k\zeta z})\right\} x$$
$$x\left\{\frac{\cos\zeta}{\zeta^2}+\frac{\sin\zeta}{\zeta}-\frac{\cos\zeta}{2}-\frac{1}{\zeta^2}\right\}J_1(\rho\zeta)d\zeta \qquad (4.3.17b)$$

$$s_{\rho\rho} = \frac{1}{\{(1+k^2)^2-4k\}}\left[\int_0^\infty \{(1+k^2)^2 e^{\xi z} - 4k^3 e^{k\xi z}\}\left\{\frac{\cos\xi}{\xi^2}+\frac{\sin\xi}{\xi}-\frac{\cos\xi}{2}-\frac{1}{\xi^2}\right\}x\right.$$
$$\left. xJ_0(\rho\xi)d\xi - \frac{2k^2}{\rho}\int_0^\infty \{(1+k^4)e^{\xi z} - 2ke^{k\xi z}\}\left\{\frac{\cos\xi}{\xi^3}+\frac{\sin\xi}{\xi^2}-\frac{\cos\xi}{2\xi}-\frac{1}{\xi^3}\right\}J_1(\rho\xi)\rho\xi\right]$$

(4.3.17c)

$$s_{\theta\theta} = \frac{1}{\{(1+k^2)^2-4k\}}\left[(1-k^4)\int_0^\infty e^{\xi z}\left\{\frac{\cos\xi}{\xi^2}+\frac{\sin\xi}{\xi}-\frac{\cos\xi}{2}-\frac{1}{\xi^2}\right\}J_0(\rho\xi)d\xi\right.$$
$$\left. +\frac{2k^2}{\rho}\int_0^\infty \{(1+k^2)e^{\xi z} - 2ke^{k\xi z}\}\left\{\frac{\cos\xi}{\xi^3}+\frac{\sin\xi}{\xi^2}-\frac{\cos\xi}{2\xi}-\frac{1}{\xi^3}\right\}J_1(\rho\xi)\rho\xi\right]$$

... (4.3.17d)

Numerical result:

The variation of the radial component of incremental stress has been shown in Fig. [3].

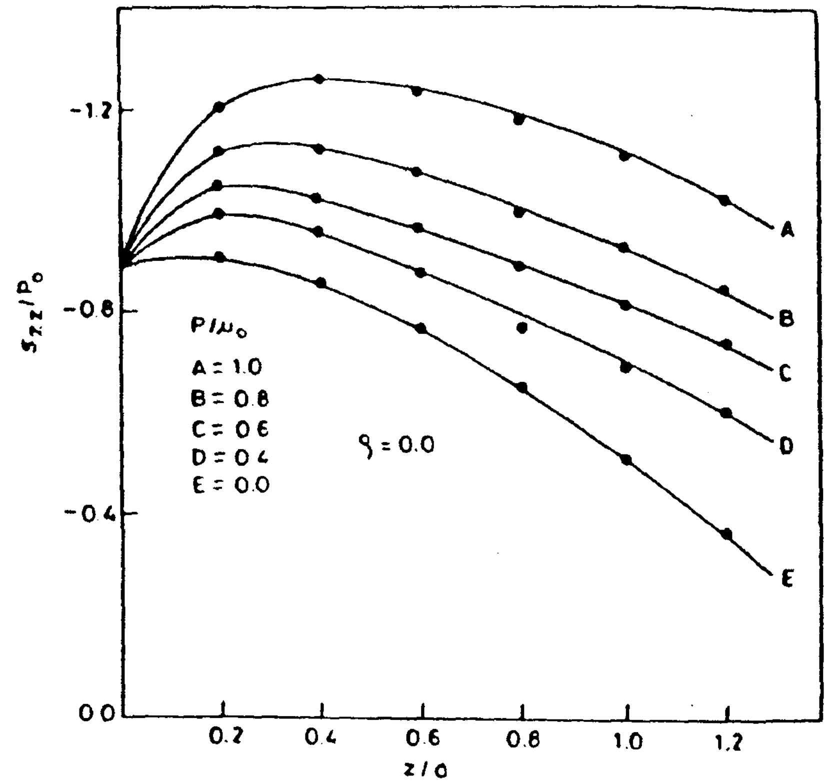

Fig. 3. The variation of the radial component of incremental stress S_{zz} with z

Chapter-5

CHAPTER – V

PUNCH PROBLEMS AND MODERN TECHNIQUES

5.1 PUNCH PROBLEMS

In preceding chapter we have discussed the distribution of stresses and strains around some linear crack in an infinite plane elastic medium. In the present chapter we propose to discuss some indentation problems or "Punch Problems". It is supposed that a perfectly rigid solid of revolution of prescribed shape, whose axis of revolution coincides with z-axis is pressed normally against the plane z = 0. of a semi-infinite elastic medium $z \geq 0$. This deformed surface of elastic medium will fit the rigid body over the part between the lowest point and circular section of radius a .It has been assumed that the shearing stress zero at all points of the boundary z=0, the z-component of surface displacement is prescribed over the region $r \leq a$, $z = 0$, and the normal stress is zero on the remaining part of the boundary. Hence, for z = 0

$$\left.\begin{array}{rl} u_z = f(r), & -a \leq r \leq a \\ \sigma_z = 0, & r > a \\ \tau_{zx} = \tau_{zy} = 0, & 0 \leq r < \infty \end{array}\right] \qquad (5.1.1).$$

Also all components of stress and displacement tend to zero $\left(r^2+z^2)^{\frac{1}{2}}\right)\to\infty$. However, if we consider a semi-inifnite plane medium occupying the region y≥0, and *i* portion ab of the boundary y=0 is under the action of a punch, then the boundary conditions are

$$\left.\begin{array}{lll} u_y & = f(x), & \text{on ab} \\ \tau_{xy} & =0, & \text{everywhere on ox xy} \\ \sigma_Y & = 0, & \text{on ox outside ab} \end{array}\right] \quad (5.1.2)$$

Green and Zeena [42] discussed the solution of the problem by the complex variable method already discussed. Green and England [40] gave an alternative method solving punch and crack problems. Though the problem of only one segment ab under the action of the punch has been discussed, it can be generalized for n segments in a similar manner. The results of indentation of half plane by a rectangular block and by a circular block have been obtained.

Recently the indentation of an elastic half plane by an infinite row of parabolic blocks has been discussed [3].These blocks are assumed to be identical, equally spaced and acting on the boundary y=0 of the half plane. The length of the region of contact of each block vith the half-plane is taken to be"2a"&nd the distance between the centers of two consecutive blocks to be d.

Following the method of Green and Zeerna [42] the components of stress and strain are given as :

$$\left.\begin{aligned}\sigma_y - i\tau_{xy} &= 2[\phi'(z) - \bar{\phi}'(\bar{z}) + (\bar{z} - z)\bar{\phi}''(\bar{z})] \\ \sigma_y + i\tau_{xy} &= 2[\phi'(z) + \bar{\phi}'(\bar{z}) + (\bar{z} - z)\bar{\phi}''(\bar{z})]\end{aligned}\right] \quad (5.1.3)$$

$$\mu(u + iv) = \alpha\phi(z) + \bar{\phi}(\bar{z}) + (\bar{z} - z)\bar{\phi}'(\bar{z}) \quad (5.1.4)$$

where & is given by the equation (2.2.11) The boundary conditions are taknen as

$$\left.\begin{aligned}&v(x,0) = \alpha_0 - \beta_o x^2, \ |x - nd| \le a \\ &\sigma_y(x,0) = 0, \ \ a \le |x - nd| \le d - a \\ &\tau_{xy}(x,0) = 0, \qquad for\ all\ x\end{aligned}\right] \quad .... (5.1.5)$$

where $\alpha_0 > 0$, $\beta_0 > 0$ and n = 0, ±1, ±2, ±3 ---- The complex stress $\phi(z)$has ultimately been found out and the expression for the normal stress σ_y obtained. For a given thrust of the length of the region of contact "*2a*" for each punch , the relation is

$$p = -\int_{-a}^{a} \sigma_y(x,o)dx \qquad (5.1.6)$$

The variation of the normal stress under a typical punch for different values of 2a/d has been shown graphically.

4.2 FINITE ELEMENT METHOD C FEM 3

All the methods to determine stress distribution in a certain ealstic medium are more or less classical and they are not convenient for problems in which stresses vary drastically. As an example we can mention the state of distribution of stresses on various parts of an aeroplane during various stages of its flight. Some parts experience tremendous pressure and some parts relatively less. The classical methods which generally give a closed type solution are unlikely to be of much help in such problems.

Although the concept of finite element method was systematically developed by workers like Clough [22],White [137].Friedriehs [32] etc.,this was being used by applied mathematicians and physicits in some form or the other. Later on the development of finite element method got momentum and it was found to be a very powerful tool for various types of problems like those in elasticity, fluid mechanics, tribology (lubrication problems) heat transfer and other field problems. These problems are generally divided into three categories:

(i)These are equilibrium problems or time independent problems. Most problems of the FEM belong to this category. In the solid mechanics area for the solution of problems, all we need is to find the displacement or stress distribution(or perhaps the temperature distribution) for a given mechanical or thermal loading. These are analogous.

(ii) There are eigen value problem or steady state problems whose solution often require the determination of natural frequency and modes of variation of solids and fluid.

(iii) These are propagation problems or time dependent problems of continuum mechanics. This category consists of problems that result when the time dimension is added to the problems of the first two categories. As already mentioned the solution of FEM comprise the following steps:

(a) Discretize the continuum.

(b) Select interpolation function for *a* typical element .

(c) Find the element properties.

(d) Assemble the element properties to obtain the system of equations.

(e) Solve the system of equations.

(f) Make additional computations if desired(post processing.

4. 3 FEM TO ELASTICITY PROBLEMS

In most applications of the FEM to solid mechanics problems, variational principle is used to derive the necessary element properties equations. The three most commonly used variational principles are the principle of minimum potential energy, the principle of complementary energy and the Reissner principle.

I. Minimum potential energy principle (Principle of virtual displacement)

We consider an elastic body which is deformed by the action of certain body forces and the surface tractions. The potential energy is defined as the strain energy minus the work done by the external forces. As given by Love [67] the theorem of minimum potential energy is as follows :

The displacement (u,v,w) which satisfies the differential equations of equilibrium as well as the conditions at the bounding surface, yields a smaller value for the potential energy than any other displacement which satisfies the same conditions at the bounding surface. Therefore, if U_1(u.v.w) is the potential energy U_2(u,v,w) is the strain energy and U_3(u.v.w) is the work done by the applied loads during displacement changes, then, according to the principle we have at equilibrium :

$$\delta U_1(u,v,w) = \delta[U_2(u,v,w) - U_3(u,v,w)] = \delta U_2(u,v,w) - \delta U_3(u,v,w) = o \qquad (5.3.1)$$

where the variation is taken with respect to the displacement while all other parameters are kept fixed. The strain energy of a linear elastic body is defined to be

$$U_2(u,v,w) = \frac{1}{2}\int\int_V\int \lfloor \varepsilon \rfloor [\sigma] dv \qquad (5.3.2)$$

with the Hooke's law

$$U_2(u,v,w) = \frac{1}{2}\int\int\int \lfloor \in \rfloor [C][\varepsilon]dv \tag{5.3.3}$$

where v is volume and

$$[\in] = [\in_x \in_y \in_z \ Y_{xy} y_{xz} Y_{yz}]^T \tag{5.3.4}$$

Following Huebner and Thornton [50] we ultimately get the general potential energy functional as :

$$U_1(u,v,w) = \frac{1}{2}\int\int_{V}\int \left[\lfloor\bar{\delta}\rfloor [B]^T \lfloor\bar{\delta}\rfloor - 2\lfloor\bar{\delta}\rfloor [B]^T [C][\in o^*]\right] dw -$$

$$-\int\int_{V}\int \lfloor F^*\rfloor [\bar{\delta}]dv - -\int\int_{S} \lfloor T^*\rfloor [\bar{\delta}]ds \tag{5.3.5}$$

The displacement components u.v.w which minimises U_1 and satisfies all the boundary conditions are the equilibrium displacement components. When using this principle in FEH, we assume the displacement components in each element and then use the functional U_1 to derive element equations. This approach is call led the displacement method or stiffness method, The compatibility conditions are identically satisfied.

II. Minimum complementary energy principle, (principle of virtual stress)

We have seen that the minimum potential energy principle corresponds to the equilibrium conditions. But the complementary energy principle corresponds to the compatibility conditions in an elastic body. The principle can be stated as follows:

"In an elastic body the state of stress which satisfies the stress- strain relations in the interior and all the given displacement boundary conditions also minimises system's complementary energy ".

If E_c (σ_x,σ_y, --- τ_{zx}) is the complementary energy, E_S (σ_x,σ_y, --- τ_{zx}) the complementary stress energy and W the work done by the applied loads then according to this principle :

$$\delta E_C = \delta E_S - \delta W = 0 \qquad (5.3.6)$$

where the variation is taken with respect to the stress components rather then displacement components. The complementary stress energy is defined as

$$U_S(\sigma_x,\sigma_y,---\tau_{zx}) = \frac{1}{2}\int\int_V\int \lfloor\sigma\rfloor[D]\{\sigma\}dv \qquad (5.3.7)$$

where V is the volume of the body, with initial strains $\{\in_o\}$

$$U_c = \frac{1}{2}\int\int_V\int [\lfloor\sigma\rfloor[D]\{\sigma\} + 2\lfloor\sigma\rfloor\{\in_o\}]dv \qquad (5.3.8)$$

The complementary work of the external load is given by

$$W = \int\int_V \lfloor T\rfloor[\delta]ds \qquad (5.3.9)$$

where T is the surface tractions and $\{\delta\}$ is the prescribed displacement matrix. Thus the complementary energy functional becomes

$$U_S(\sigma_x,\sigma_y,---\tau_{zx}) = \frac{1}{2}\int \int_V \int [\lfloor\sigma\rfloor[D]\{\sigma\} + 2\lfloor\sigma\rfloor\{\in_o\}]dv \quad (5.3.10)$$

When this principle is used in FEM, we assume the form of the stress field in each element. This approach is called the force method or the flexibility method and the element equations are identically satisfied. III. Reissners principle in this principle variations of both displacement and stress are considered. This principle can be called to be a combination of both the previous principles. The principle states that

$$\delta E(u,v,w,\sigma_x,-\sigma_y-\tau_{zx}) = o$$

where E is the functional for a linear elastic material and is given by

$$E = \int \int_V \int \left[\lfloor\sigma\rfloor\{\in\} - \frac{1}{2}\lfloor\sigma\rfloor[D]\{\sigma\} - \lfloor\bar{\delta}\rfloor\{F\}\right]dv - \int_{S_1}\int \lfloor\bar{\delta}\rfloor\{T\}ds - \int_{S_2}\int \lfloor\delta-\delta_o\rfloor\{T\}ds \quad (5.3.11)$$

The variation of arguments does not give external values but only a stationary value. When Reissner's principle is used in FEH, then forms of both displacement and stress fields are assumed in each element. The above mentioned principles are must commonly used. However, there are other variation principles like Hamilton's minimum principle etc.

4.4 SOME RECENT WORK

In the preceding chapter we have discussed some punch and crack problems. The problems of stiffening and inclusion are also of much interest in solid mechanics. Recently in 1993, Ishikawa and Kohno [53] have analyzed the stress fintensity factor of an interface crack of a rectangular rigid inclusion. The problem has been considered as plane elastic and the inclusion has been assumed to be completely bonded to the interior of an elastic infinite medium, except of a portion which is regarded as an interface crack. Muskhelisvili's stress function is determined from terms of finite series of the function for conformal mapping, by which the inclusion is mappec onto the unit circle. The stress intensity factors for the interface crack are than determined under the equal biaxial loading condition.

Many authors like Matos et.al. [77],Sun and Jili [125] etc .have analyzed stress intensity factor and energy release rates of the interface cracks between the circular or elliptical inclusion and the matrix for the models of the interface crack of short fibre which reinforced composites. For more exact models of the short fibres which reinforce the matrix, the interface crack between the hypotrochoid inclusion and the matrix was analyzed by Malyshev and Salganik [71]. However, there is great difference between the hypotrochoid shape and that of the actual short fibre, because the angle of the corner of the hypotrochoid shape is zero where as the corner of the actual short fibers has a finite angle. Due to this consideration Ishikawa and Kohno have adopted a rectangular rigid inclusion and two dimensional elastic analysis of an interface crack between a rectangular rigid inclusion and the matrix is performed using the method of Muskhelishvili [94].They have analyzed two types of interface cracks :

(a) The crack which is located on the side of the short fibre end i.e. on the short side of a rectangular rigid inclusion.

(b) The crack which is extended to the longitudinal sides of the short fibre i.e. extended from the short side to the long side of a rectangular rigid inclusion.

Using the method developed by Muskhelishvili [94], the rectangular inclusion is supposed to be embedded in the infinite plate i.e. xy-plane. By Schwatz-Christoffel transformation

$$z = w\ (\zeta)$$

$$\frac{d_z}{d\xi} = \frac{aR}{\zeta^2} \Pi^4_{i=1} (\zeta_p - \zeta)^{\frac{ap}{\Pi}} \qquad (5.4.2)$$

The outer region of the rectangle in the xy-plane is mapped onto outer region of the unit circle ξ in the ξplane. Here a is R half length of a diagonal of the rectangular inclusion and R is a real constant. The points $\xi p(\rho=1,2,3,4)$ on the that correspond to the four corner points of a rectangle on the xy-plane are given as

$$\left.\begin{array}{l} \xi = e^{i.o} = 1, \xi_2 = e^{i\ln}, \xi_3 = e^{in} = -1 \\ \\ \xi_4 = e^{i(n+\ln)} = -e^{i\ln} \end{array}\right] \quad \ldots (5.4.3)$$

The shape of rectangle is determined by the value of 1. In the polar coordinates (r,θ) in the ξ-plane, the stress and the derivative form of the displacement are shown by the complex stress function ϕ (ζ) and ψ (ζ) as

$$\sigma_r + \sigma_\theta = 2[\phi(\xi) + \overline{\phi(\xi)}\ ]$$

$$\sigma_r + i\tau_{r\theta} = \phi(\xi) + \overline{\phi(\xi)} - \frac{\bar{\xi}}{\xi w'(\zeta)}\left[w(\xi)\overline{\phi'(\zeta)} + w'(\zeta)\psi(\zeta)\right] \quad (5.4.4)$$

$$2G\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) = i\zeta w'(\xi)\left[\alpha\phi(\zeta) - \overline{\phi(\zeta)}\right] + i\bar{\zeta}\left[\overline{w(\zeta)\phi'(\zeta) + w'(\zeta)\psi(\zeta)}\right]$$

(5.4.5)

where G is the modulus of rigidity and ∞, the equivalent poission ratio defined by the equation (2.2.11).The boundary conditions of the interface between inclusion and the matrix has been assumed as

$$\sigma_r + \tau_{r\theta} = o,\ on\ L \quad (5.4.6)$$

$$u + iv = o, \quad on\ L, \quad (5.4.7)$$

The equation (5.4.5) represents the interface crack and (5.4.6) indicates that the inclusion is perfectly bonded with the matrix. The solution of the problem has been obtained in the form

$$F(\zeta) = P(\zeta)x(\zeta) \tag{5.4.8}$$

Where

$$\left.\begin{aligned} &F(\zeta) = w'(\zeta)\psi(\zeta) \\ &P(\zeta) = \sum_{n=Nm}^{1} d^n \zeta^n \\ &X(\zeta) = (\zeta - f)^{-1/1+2\eta}(\zeta - e)^{-\frac{1}{2}-in} \\ &\eta = \log\alpha / 2\pi \end{aligned}\right]$$

Further, they have obtained the stress intensity factor of an inter crack on the short side of a rectangular rigid inclusion and then of an inter crack extended to the long side of a rectangular rigid inclusion.

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